Trends in Mathematics

Piotr Kielanowski
Anatol Odzijewicz
Emma Previato
Editors

Geometric Methods in Physics XXXVII

Workshop and Summer School, Białowieża, Poland, 2018

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Trends in Mathematics

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Preface

This volume contains a selection of papers presented during the Thirty-Seventh *Workshop on Geometric Methods in Physics* in 2018, organized by the Institute of Mathematics of the University of Białystok. About 70 physicists and mathematicians from important scientific centers from all over the world attended the workshop. The Workshop was accompanied by the School of Geometry and Physics, where several cycles of didactic lectures on important and new subjects for advanced students and young scientists were presented. Abstracts of these lectures are also included in this volume. Information on previous and upcoming schools and workshops, and related materials, can be found at the URL: http://wgmp.uwb.edu.pl.

The geometrical methods in Physics constitute a very wide branch of mathematical physics. The main topics that have been discussed this year are: quantum groups, non-commutative geometry, integrable systems, differential equations, operator algebras, quantization and infinitely dimensional geometry.

An important event during the workshop was a session dedicated to the scientific activity of professor Daniel Sternheimer on the occasion of his 80th birthday.

Białowieża, which is a traditional place, where the Workshops take place deserves special mention. It is located on the border between Poland and Belarus and is the only place in Europe where there are the remains of primeval forests and was designated a UNESCO World Heritage Place. Such close contact with nature creates a special atmosphere during the workshop and scientific discussions take less formal character.

The Organizing Committee of the 2018 Workshop on Geometric Method in Physics gratefully acknowledges the financial support of the University of Białystok.

The Editors

Participants of the XXXVII WGMP

(*Photo by Tomasz Goliński*)

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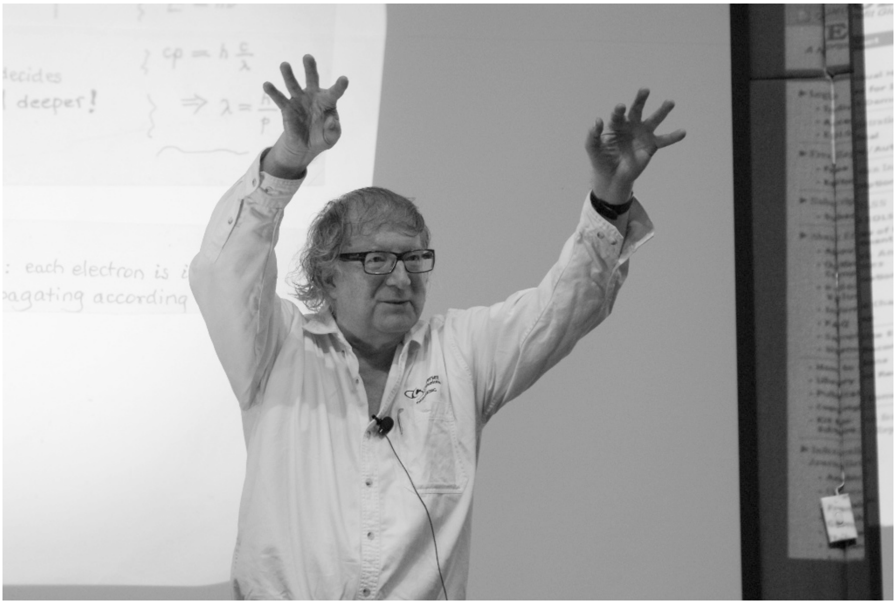
In Memoriam Bogdan Mielnik



(*Photo by Alonso Contreras Astorga*)

Bogdan Mielnik 1936–2019

Our colleague and friend Bogdan Mielnik sadly passed away in Mexico City on January 22, 2019, due to complications of heart surgery.

Bogdan was born on May 6, 1936, in Warsaw, Poland. He was a brilliant scientist whose career developed mainly at the Institute of Theoretical Physics, Warsaw University (Poland) and at the Center for Research and Advanced Studies (Cinvestav) in Mexico.

He made many important contributions to quantum theory. In the 1960s and 1970s Bogdan was one of the pioneers in using geometric methods in quantum mechanics, in particular theories of filters and convex sets which now are standard tools in quantum information science. During the 1980s he became one of the leaders of the factorization method, a simple technique for generating new exactly solvable potentials and for implementing spectral design. His work on quantum control and dynamical manipulation was equally relevant, including such suggestive proposals as the idea of reversing the time evolution of a system and the introduction of evolution loops as the basis for inducing arbitrary unitary transformations on quantum systems.

Above all, Bogdan was an exceptionally wise man with a highly individual way of thinking. He did not publish many papers, and was critical of massive publications. Each of his works was deeply thought-out and based on profoundly original elements. Thanks to this, he came to be recognized as one of the great authorities in the foundations and formalism of quantum mechanics.

He loved to interact with people, especially with students. This led to his creating an entire school of thought in Theoretical and Mathematical Physics with his many excellent students in Mexico and in Poland. One of Bogdan's former students – Anatol Odzijewicz – has been the main organizer of the Białowieża Workshops. Bogdan participated enthusiastically in these Workshops during recent years, making important contributions to the program and giving original talks.

Bogdan also had a great sense of humor, a unique aesthetic taste in the visual arts, and a great literary talent. He loved science fiction, and wrote brilliant stories himself within this genre. We will always remember his lively imagination, his acute criticism, and his intellectual irreverence – the qualities that made him the person he was. We shall miss him greatly.

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Some aspects of the work of Daniel Sternheimer

Giuseppe Dito

Mathematics Subject Classification (2000). 01A70, 53D55, 81R05, 22E60.

Keywords. Unification, symmetry, Lie algebras, representations, integrability, quantization, deformations.

A life dedicated to mathematical physics

Daniel Sternheimer was born in 1938 in Lyon, France. After graduating in mathematics from the University of Lyon in 1958, he temporarily moved to Israel where he earned his Master's degree in 1961 from the Hebrew University of Jerusalem. It was in Jerusalem that Daniel met his lifetime chief collaborator and close friend, Moshé Flato. Back to France, he prepared his D.Sc. thesis and received his *Doctorat d'État* from the University of Paris in 1968 with a thesis titled *Quelques problèmes concernant les algèbres de Lie posés par la physique mathématique. Croissance de séries de Dirichlet.*

Daniel pursued his entire career at the C.N.R.S. where he was recruited as an *Attaché de recherches* as early as he started to work on his D.Sc. thesis in 1961. He has been pursuing his career at C.N.R.S. until his retirement in 2003, climbing all the steps leading him to the level of Research Professor. Since then, he has been sharing his time between France, Israel, Japan and the U.S.A. Daniel held several visiting positions including one as a Visiting Professor at Keio University in Yokohama for the period 2004-2010. Since 2002 he is a member of the board of governors of the Ben-Gurion University of the Negev. In 2004 he became an honorary professor of Saint Petersburg University

He is currently a visiting research fellow at Rikkyo University in Tokyo and an associate member of the Mathematics Institute in Dijon.

A considerable time and energy was devoted by Daniel in serving the community. He was involved in the International Association of Mathematical Physics and served as a member of its Executive Committee (1978-1984). He was also a member of the Mathematical Physics Commission of the International Union of Pure and Applied Physics for the period 1981-1987.

He has served on the editorial board of several journals including *Reports on Mathematical Physics*, *Reviews in Mathematical Physics*, and especially *Letters in Mathematical Physics* (LMP). In fact Daniel was involved in LMP from the very beginning in the mid-seventies, when it was founded by M. Flato with the help of M. Guenin, R. Rączka and S. Ulam. After the passing of M. Flato in 1998, Daniel took the direction of the journal and thus ensured continuity. He is still an active editor of LMP.

Major achievements

In what follows, I made a deliberate choice to highlight 3 topics on which Daniel has worked that I found particularly important, stimulating and beautiful, and also that had (and have) a great impact in mathematical physics. These subjects are presented chronologically.

Unification of symmetries. Symmetries in Nature and in elementary particle physics are at the heart of Daniel's work. This topic attracted a lot of attention in the sixties and it was a natural playground for him. In 1965, L. O'Raifeartaigh published a paper [16] in which it was advocated that any unitary representation of a Lie algebra containing the Poincaré Lie algebra as a subalgebra does not generate a splitting of the spectrum of the mass operator, namely, the extra symmetries would lead to degenerate multiplets in the mass spectrum. This no-go "theorem" was carefully analyzed from a mathematical viewpoint by Daniel in a joint paper with M. Flato [6]. They showed that an actual proof was not given in [16] and expressed doubts about the validity of the claimed statement. Actually, an explicit counterexample was given [7] by considering a representation of the Lie algebra of the conformal group, $\mathfrak{su}(2,2)$, which cannot be integrated to a unitary representation of the conformal group due to the lack of analytic vectors (see below) in the Lie algebra representation. Another counterexample involving an infinite dimensional Lie algebra and discrete mass spectrum appeared in [8]. Daniel has studied further mathematical aspects of these questions in [17, 18].

The general context of this period is vividly reviewed by Daniel in his contribution [19] to the WGMP XXXII volume.

Integrability of representations. Recall that an analytic vector for a densely-defined operator T on a Banach space B is a vector $x \in B$ belonging to the domain of all the T^n, $n \in \mathbb{N}$, such the series $\sum_n \frac{t^n}{n!}\|T^n x\|$ has a positive radius of convergence. This notion was introduced in a classical paper by E. Nelson [15] and used by him to provide a criterion for the integrability of a representation of a finite-dimensional Lie algebra by skew-symmetric operators to a unitary representation of the corresponding Lie group. More precisely, Nelson's criterion says that: given a representation ρ of a real finite-dimensional Lie algebra $\mathfrak{g}$ in a Hilbert space H by skew-symmetric operators $\rho(g)$, $g \in \mathfrak{g}$, on a common dense $\mathfrak{g}$-invariant domain $D \subset H$ and such that the Laplacian in a given basis of $\mathfrak{g}$: $\Delta_\rho := \sum_i \rho(X_i)^2$ is essentially self-adjoint, then ρ exponentiates to a unitary representation of the corresponding (connected and simply connected) Lie group.

In an important paper [9], M. Flato, J. Simon, H. Snellman, and D. Sternheimer have weakened the assumptions in Nelson's criterion by providing a new integrability criterion, nowadays known as the FS -criterion, that we now state:

Given a representation ρ of a real finite-dimensional Lie algebra $\mathfrak{g}$ in a Hilbert space H by skew-symmetric operators $\rho(g)$, g $\mathfrak{g}$, on a common dense invariant domain $D \subset H$ such that all of the vectors of D are analytic vectors for the operators $\rho(X_i)$ in a given basis $(X_i)_i$ of $\mathfrak{g}$, then ρ exponentiates to a unitary representation of the corresponding Lie group.

The advantage of the FS -criterion is also a practical one: it does not require joint analyticity as in Nelson's criterion, but only separate analyticity, i.e. for each generators and not for the whole Lie algebra.

Deformation quantization. This is by far the field in which Daniel's contributions are the best known.

The general theory of deformations of algebraic structures (e.g. Lie or associative) was introduced in the sixties and has been studied over a decade by M. Gerstenhaber in a series of papers [12, 13]. Around the mid-seventies, with M. Flato and A. Lichnerowicz, Daniel started to study deformations of the Poisson bracket on a symplectic manifold [10,11]. The idea was to deform the Poisson bracket $\{\cdot,\cdot\}$ of classical mechanics into a bracket

$$[f,g]_\lambda = \{f,g\} + \sum_{k\geq 1} \lambda^k P_k(f,g),$$

where f, g are smooth functions on the symplectic manifold. The deformed bracket is required to formally satisfy Jacobi identity and the P_k are skew-symmetric bidifferential operators of order at most 1 in each argument (one speaks of 1-differentiable deformations). It defines a new Lie structure on the space of formal series in a parameter λ with coefficients in the space of smooth functions on the symplectic manifold (actually contact manifolds were also considered). A thorough study of the cohomology induced by these deformations was done in [11]. Although this paper was of purely mathematical nature, the idea of its potential applications to quantization problems was already mentioned in the introduction.

In an important work [21], J. Vey extended the context of [10] to deformations of the Poisson brackets where the order of the P_k's is not bounded, and as a by-product rediscovered the old Moyal product and bracket and, showed the existence of such deformations on any symplectic manifold with vanishing third Betti number.

The crucial step was announced in [1]: an autonomous approach to Quantum Theory (i.e. without Hilbert space) is put forward by providing an interpretation of quantum mechanics as a deformation of classical mechanics. The Planck constant is interpreted in this context as the deformation parameter. The notion of star-product was introduced and thoroughly studied in the comprehensive and very influential twin papers [2, 3] published in 1978 in *Annals of Physics*. One should stress that many of the standard examples of Quantum Mechanics such as harmonic oscillators, hydrogen atom, etc., have been thoroughly treated in [3]. The key notion here is that of star-exponential of the Hamiltonian which allows for a *formal* spectral theory. Needless to say that the results are in agreement with traditional Quantum Mechanics.

Since then, deformation quantization has become a classical tool in many other fields of physics and mathematics, and is taught in graduate programs worldwide. It has triggered so many works that it is impossible to even mention them in this short Note. I shall confine myself to the beautiful construction of B.V. Fedosov [5] and the Formality Theorem proved in 1997 by M. Kontsevich [14] which, among its many important consequences, implies that any smooth Poisson manifold admits a deformation quantization.

For a review with many historical remarks on deformation quantization, the reader can consult with benefit the extensive review by Daniel [20] (see also [4] for a more recent review).

There is still a lot to say about the contributions of Daniel to other fields such as singletons and conformal invariance, non linear representations of Lie groups and algebras, quantum groups, etc. All these themes are still at the center of Daniel's scientific and philosophical thoughts about the role of symmetries and deformations in modern mathematical physics (see e.g. [19]).

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Daniel Sternheimer is 80 years young!

(Photo by Tomasz Goliński)

Part I

Differential equations and integrable systems

Geometric Methods in Physics. XXXVII Workshop 2018
Trends in Mathematics, 3–12

On canonical parametrization of phase spaces of Isomonodromic Deformation Equations

Mikhail V. Babich

Abstract. The space of the Fuchsian systems is the algebraic Poisson manifold, and the equations of the Isomonodromic Deformations are the Hamiltonian equations. The internal symmetry of the problem makes it possible to reduce the dimension of the problem using the symplectic-quotient theory. The phase-space is constructed from the orbits of (co)adjoint representation of the general linear group. The presented parametrisation of the quotient-space is based on the construction of the flag coordinates on the orbits. The simplest non-trivial case that is Painlevé VI case is considered as an example.

Mathematics Subject Classification (2000). Primary 34-02, 34M55; Secondary 33E17, 34A26, 34A30, 34M35, 37J05.

Keywords. Isomonodromic deformations, Fuchsian equations, flag coordinates, momentum map, Painlevé VI equation.

1. Fuchsian systems and their monodromy

A differential system is called Fuchsian if the coefficients have simple poles only:

$$\frac{d}{dz}\overrightarrow{\psi} = A(z)\overrightarrow{\psi} = \sum_{k=1}^{N-1} \frac{A^{(k)}}{z - z_k}\overrightarrow{\psi}, \quad A^k \quad \mathrm{gl}(n,\mathbb{C}),\ z, z_k \quad \mathbb{C},\ \ \overrightarrow{\psi} = \overrightarrow{\psi}(z) \quad \mathbb{C}^n.$$

We denote $\sum_{k=1}^{N-1} A^{(k)}$ by $-A^N$, so $\sum_k A^{(k)} = 0$ and $z_N := \infty \quad \overline{\mathbb{C}}$. Let us consider the fundamental group π_1 of $\overline{\mathbb{C}} \setminus \{z_1, \dots, z_N\}$. It can be treated as a group of loops passing some fixed point $P_0 \quad \overline{\mathbb{C}} \setminus \{z_1, \dots, z_N\}$. The group operation is the sequential passing the loops.

Let us consider a fundamental system of solutions $\overrightarrow{\psi}_1, \dots, \overrightarrow{\psi}_n$, and collect them to a square matrix $\Psi := \big(\overrightarrow{\psi}_1 \overrightarrow{\psi}_2 \dots \overrightarrow{\psi}_n\big)$. It is evident that $\Psi \quad \mathrm{GL}(n,\mathbb{C})$.

We get two fundamental solutions $\Psi(z)$ and $\Psi\,|_l(z)$ of the same system if we denote by $\Psi\,|_l(z)$ the same matrix-solution $\Psi(z)$ but after passing the loop l. Two fundamental solutions of the same system differ by a constant right factor that we denote by M_l:

$$\Psi\,|_l(z) = \Psi(z) M_l\ , \quad M_l \quad \mathrm{GL}(n, \mathbb{C}).$$

It is not difficult to see that such analytical continuation of Ψ along the loops induces the (anti)representation of the fundamental group

$$\pi_1\big(\overline{\mathbb{C}} \setminus \{z_1, \dots, z_N\}\big) \to \mathrm{GL}(n, \mathbb{C}).$$

The question: "How can we change the differential system to keep this representation constant?" is called *an Isomonodromic Deformation problem.*

There are some trivial transformations like the Möbius transformation of $z_k \to \frac{az_k+b}{cz_k+d}$ with constant $A^{(k)}$'s, or the simultaneous conjugation of all $A^{(k)}$ by the same matrix $A^{(k)} \to g^{-1}A^{(k)}g$. Nevertheless there are nontrivial transformations too.

Let us introduce a deformation parameter t, so $A^{(k)} = A^{(k)}(t)$ and $z_k = z_k(t)$. It is the classical result (see [3]), that the Isomonodromy condition of $d\Psi = A(z,t)dz$ is equivalent to the existence of a such matrix-function $B(x,t)$ that a form $\omega := Adz + Bdt$ is flat:

$$d\omega = \omega \wedge \omega.$$

Such form is called *a deformation form.*

2. Isomonodromic Deformation Equations. Schlesinger system and its phase space

There is a following ansatz that gives the most important family of the isomonodromic deformations:

$$\omega = \sum_k A^{(k)} \frac{dz - dz_k}{z - z_k},$$

it is a so-called *Schlesinger ansatz.* The deformation parameter is the position of the poles here, for example $z \ =: t, dz_k = 0, k \neq 3$.

The flatness condition is equivalent to the nice system of equations on the residues $A^{(k)} = A^{(k)}(z_1, z_2, \dots)$:

$$dA^{(k)} + \left[A^{(k)}, \sum_{i \neq k} A^{(i)} \frac{dz_k - dz_i}{z_k - z_i}\right] = 0,$$

they are the so-called Schlesinger equations.

Schlesinger equations define a dynamical system on the Poisson space

$$\mathrm{gl}(n, \mathbb{C}) \times \cdots \times \mathrm{gl}(n, \mathbb{C}) \ni \big\{A^{(1)}, A^{(2)}, \dots, A^{(N)}\big\}.$$

It can be easily verified that it is Hamiltonian system. The Hamiltonians are generated by the differential

$$h := \sum_{i<j} \mathrm{tr}\ A^{(i)} A^{(j)} \frac{dz_i - dz_j}{z_i - z_j}.$$

Namely, the evolution of the system according to "time" z_i is Hamiltonian and the corresponding Hamiltonian is $h\,\big|_{dz_j=0}/dz_i$.

The Schlesinger equations do not change the conjugation classes of the matrices $A^{(k)}$, because the equation on the matrices has the form $dA^{(k)} = [A^{(k)}, \dots]$. Let us denote by $\mathcal{O}$ the set of all matrices similar to some J,

$$\mathcal{O} := \bigcup_{g \in \mathrm{GL}(n,\mathbb{C})} g^{-1} J g.$$

It is the orbit of (co)adjoint representation of $\mathrm{GL}(n, \mathbb{C})$. Each matrix $A^{(k)}$ belongs to a fixed orbit. We denote this orbit by $\mathcal{O}^{(k)} \ni A^{(k)}$.

A simultaneous conjugation of the elements of the set

$$\left\{A^{(k)}\right\}_{k=1}^{N} \to \left\{g^{-1} A^{(k)} g\right\}_{k=1}^{N}$$

by any matrix g $\quad\mathrm{GL}(n, \mathbb{C})$ moves the flow on $\mathcal{O}^{(1)} \times \mathcal{O}^{(2)} \times \cdots \times \mathcal{O}^{(N)}$ to itself:

$$d\big(g^{-1} A^{(k)} g\big) = -\left[g^{-1} A^{(k)} g, \sum_{i \neq k} g^{-1} A^{(i)} g \frac{dz_k - dz_i}{z_k - z_i} \right].$$

It is the symmetry of the Hamiltonian system and the flow can be projected on the quotient space

$$\mathcal{O}^{(1)} \times \mathcal{O}^{(2)} \times \cdots \times \mathcal{O}^{(N)} / \mathrm{GL}(n, \mathbb{C}).$$

The same projection have all the equations that differ from the presented one by the gauge transformations with non-constant $g = g(z_1, \dots, z_N)$:

$$dA^{(k)} = -\left[A^{(k)}, \widetilde{\omega}_g + \sum_{i \neq k} A^{(i)} \frac{dz_k - dz_i}{z_k - z_i} \right],$$

where $\widetilde{\omega}_g$ is an arbitrary flat 1-form. It is natural to investigate just the projection that is a Hamiltonian flow on the quotient-space. It is a subject of the *symplectic reduction theory*.

The symplectic reduction theory states that the symplectic leave of such quotient-space is the constant level of the so called *momentum map*.

A momentum map is such a map

$$\mathcal{O}^{(1)} \times \mathcal{O}^{(2)} \times \cdots \times \mathcal{O}^{(N)} \to \mathrm{gl}^*(n, \mathbb{C})$$

that its value (it is a function on $\mathrm{gl}(n, \mathbb{C})$) generates Hamiltonians that produce the flows corresponding the elements of Lie algebra $\mathrm{gl}(n, \mathbb{C})$ acting on $\mathcal{O}^{(1)} \times \mathcal{O}^{(2)} \times \cdots \times \mathcal{O}^{(N)}$.

In our case the action is the diagonal action, consequently each element $G \quad \mathrm{gl}(n,\mathbb{C})$ generates the flow $[A^{(k)},G]$ on each Cartesian factor of the product, consequently the element of $\mathrm{gl}^*(n,\mathbb{C})$ in question is

$$\sum_{k=1}^{N} A^{(k)} \quad \mathrm{gl}(n,\mathbb{C}) \simeq \mathrm{gl}^*(n,\mathbb{C}),$$

because each $A^{(k)} \quad \mathrm{gl}(n,\mathbb{C}) \simeq \mathrm{gl}^*(n,\mathbb{C})$ acts on its own Cartesian factor as Hamiltonian (function) in this sum. We are interested in its zero-value level $\sum_{k=1}^{N} A^{(k)} = 0$, because it is the total residue of the differential $A(z)dz$.

A symplectic manifold $\mathcal{O}^{(1)} \times \mathcal{O}^{(2)} \times \cdots \times \mathcal{O}^{(N)} /\!\!/ \mathrm{GL}(n,\mathbb{C})$, that is by the definition

$$\left(\left\{A^{(1)}, A^{(2)}, \ldots, A^{(N)} : \sum_{k=1}^{N} A^{(k)} = 0\right\} \cap \mathcal{O}^{(1)} \times \mathcal{O}^{(2)} \times \cdots \times \mathcal{O}^{(N)}\right) / \mathrm{GL}(n,\mathbb{C}),$$

forms the phase space of the Isomonodromic Deformation Equations. It is the space that will be canonically parameterized in the present article.

3. Flag coordinates on orbit $\mathcal{O}$

In the works [1,2] flag coordinates on the coadjoint orbits of the general linear group were introduced. The method of their construction is based on the observation that the representation of matrix A from the orbit:

$$A = \begin{pmatrix} \mathrm{I} & 0 \\ Q & I \end{pmatrix} \begin{pmatrix} \lambda & P \\ 0 & \widetilde{A} \end{pmatrix} \begin{pmatrix} \mathrm{I} & 0 \\ Q & I \end{pmatrix}^{-1}$$

produces the skew-orthogonal with respect to Lie–Poisson–Kirillov–Kostant structure $\{,\}_{LP}$ families of functions on the orbit:

$$\{P_{ij}, P_{kl}\}_{LP} = \{Q_{ij}, Q_{kl}\}_{LP} = \{P_{ij}, \widetilde{A}_{kl}\}_{LP} = \{Q_{ij}, \widetilde{A}_{kl}\}_{LP} = 0.$$

Matrix $\widetilde{A}$ belongs to the orbit of the smaller dimension than the dimension of A, that makes possible to organize the iteration process.

The geometrical interpretation of the flight of the iteration is the projection of the action of $A \quad \mathrm{End}\,(\mathbb{C}^n)$ along the eigenspace corresponding to λ on a coordinate subspace. The projection induce $\widetilde{A} \quad \mathrm{End}\left(\mathbb{C}^{\widetilde{n}}\right)$, and the pair P, Q.

Let us construct Q. Consider the projection of a subset $\mathbf{e}'$ of the set $(\mathbf{e}', \mathbf{e}'')$ of the basic vectors on the eigenspace corresponding to the eigenvalue λ parallel to the coordinate subspace $\mathscr{L}(\mathbf{e}'')$ and then project a result to $\mathscr{L}(\mathbf{e}'')$ parallel to $\mathscr{L}(\mathbf{e}')$. It gives matrix Q:

$$\mathbf{e}' \to \mathbf{e}' + \mathbf{e}''Q \to \mathbf{e}''Q.$$

The projection on the eigenspace and the subsequent projection on $\mathscr{L}(\mathbf{e}'')$ can be treated as a linear map $\mathcal{Q} \quad \mathrm{Hom}\big(\mathscr{L}(\mathbf{e}'), \mathscr{L}(\mathbf{e}'')\big)$. The family of the conjugated functions $(P)_{ij}$ can be treated as coordinates coming from the opposite direction map $\mathcal{P}$ $\operatorname{Hom}\big(\mathscr{L}(\mathbf{e}''), \mathscr{L}(\mathbf{e}')\big)$:

$$A = \begin{pmatrix} * & P \\ * & * \end{pmatrix}.$$

The pairing tr $\mathcal{P}\mathcal{Q}$ coincides with the pairing of functions on the orbit generated by the Lie–Poisson–Kirillov–Kostant structure.

The transposed equality

$$A = \begin{pmatrix} \mathrm{I} & -Q_b \\ 0 & I \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ P_b & \widetilde{B} \end{pmatrix} \begin{pmatrix} \mathrm{I} & -Q_b \\ 0 & I \end{pmatrix}^{-1}$$

produces another set of functions P_b, Q_b on the orbit. It is evident that they are conjugated too. The coordinates P, Q and P_b, Q_b are called *flag coordinates* on the orbit.

Geometrically the construction of P_b, Q_b can be treated as the contraction of A on the co-eigenspace $\mathrm{im}(A - \lambda \mathrm{I})$ and the corresponding transformations of the coordinate subspaces give P_b, Q_b.

The geometrical interpretations are fundamental for the parametrization of the phase space of the Isomonodromic Deformation Equations that is the reduced space

$$\begin{cases} \mathcal{O}^{(1)} \times \cdots \times \mathcal{O}^{(N)} / \mathrm{GL}(n, \mathbb{C}), \\ \sum_{k=1}^{N} A^{(k)} = A^{\Sigma} = \text{const}. \end{cases} \tag{1}$$

Consider a case of a general position when matrices $A^{(k)}$ have one-dimensional eigenspaces only. The iteration process of the construction of the flag coordinates factorize each $A^{(k)}$ to the product of triangular matrices $Q^{(k)} \widetilde{P}^{(k)} (Q^{(k)})^{-1}$:

$$Q^{(k)} = \begin{pmatrix} 1 & 0 \\ * & \mathrm{I} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ Q_{2,1} & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ Q_{n-1,1} & Q_{n-1,2} & \dots & 1 & 0 \\ Q_{n,1} & Q_{n,2} & \dots & Q_{n,n-1} & 1 \end{pmatrix},$$

$$\widetilde{P}^{(k)} = \begin{pmatrix} \lambda_1^{(k)} & * \\ 0 & \Lambda^{(k)} \end{pmatrix} = \begin{pmatrix} \lambda_1^{(k)} & \widetilde{P}_{1,2} & \dots & \widetilde{P}_{1,n-1} & \widetilde{P}_{1,n} \\ 0 & \lambda_2^{(k)} & \dots & \widetilde{P}_{2,n-1} & \widetilde{P}_{2,n} \\ 0 & 0 & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda_{n-1}^{(k)} & \widetilde{P}_{n-1,n} \\ 0 & 0 & \dots & 0 & \lambda_n^{(k)} \end{pmatrix}.$$

We sometimes skip the upper-index "(k)" for short.

Matrix elements Q_{ij} are just canonical coordinates, but $\widetilde{P}_{ji}$ are not. To introduce coordinates P_{ji} conjugated to Q_{ij}, let us denote a square lower-right $j \times j$

block of Q by $Q_{j\times j}$:

$$Q_{j\times j} := \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ Q_{n-j+2,n-j+1} & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ Q_{n-1,n-j+1} & Q_{n-1,n-j+2} & \dots & 1 & 0 \\ Q_{n,n-j+1} & Q_{n,n-j+2} & \dots & Q_{n,n-1} & 1 \end{pmatrix},$$

and denote a non-trivial part of j's row of $\widetilde{P}$ by $\widetilde{P}_j$:

$$\big(0,\dots,0,\lambda_j^{(k)},\widetilde{P}_{j,j+1},\dots,\widetilde{P}_{j,n-1},\widetilde{P}_{j,n}\big) = \big(\overrightarrow{0},\lambda_j^{(k)},\widetilde{P}_j\big),$$

it is $(n-j)$-dimensional vector-row. A set of the canonical coordinates

$$\big(P_{j,j+1},\dots,P_{j,n-1},P_{j,n}\big) =: P_j$$

is the product

$$\widetilde{P}_j Q^{-1}_{n-j\times n-j} := P_j, \quad P_{n-1,n} = \widetilde{P}_{n-1,n} = P_{n-1} = \widetilde{P}_{n-1} \quad \mathbb{C}.$$

We can see that $P_{ji}^{(k)}$ is linear with respect to $\widetilde{P}_{ji}^{(k)}$, and $P_{ji}^{(k)} = \widetilde{P}_{ji}^{(k)}$ if $Q^{(k)} = \mathrm{I}$.

4. Parameterization of reduced space

Our plan is to parameterise the orbits and present such set of Q-coordinates that their common zero-value gives a section of

$$\mathcal{O}^{(1)}\times\mathcal{O}^{(2)}\times\dots\times\mathcal{O}^{(N)} \to \mathcal{O}^{(1)}\times\mathcal{O}^{(2)}\times\dots\times\mathcal{O}^{(N)}/\mathrm{GL}(n,\mathbb{C}),$$

consequently the rest set of the coordinates form the coordinates on the base that is the quotient manifold.

We will explicitly solve the constant momentum level equation $\sum_k A^{(k)} = 0 \ni \mathrm{gl}(n,\mathbb{C})$ after that. It will be linear equations in the terms of the flag coordinates.

4.1. Factorization with respect to adjoint $\mathrm{GL}(n,\mathbb{C})$-action

Let $A^{(N)}$ has different eigenvalues $\lambda_1^{(N)},\lambda_2^{(N)},\dots,\lambda_n^{(N)}$ for the simplicity. A set of the subspaces

$$F_s = \ker\big(A^{(N)}-\lambda_1^{(N)}\mathrm{I}\big)\big(A^{(N)}-\lambda_2^{(N)}\mathrm{I}\big)\cdots\big(A^{(N)}-\lambda_s^{(N)}\mathrm{I}\big), \quad s = 1,2,\dots,n$$

form a compleat flag $0\subset F_1\subset\dots\subset F_{n-1}$ in $\mathbb{C}^n$.

Let us denote the basic vectors by $\mathbf{e}_k$. It is evident that $Q^{(N)} = \mathrm{I}$ is equivalent to the coinciding the coordinate subspaces and the subspaces of the flag:

$$F_s = \mathscr{L}(\mathbf{e}_1,\mathbf{e}_2,\dots,\mathbf{e}_s).$$

Let us transpose the equalities now, but keep the notations: $A^{(N-1)} = Q\widetilde{P}Q^{-1}$,

$$Q = \begin{pmatrix} 1 & Q_{1,2} & \dots & Q_{1,n-1} & Q_{1,n} \\ 0 & 1 & \dots & Q_{2,n-1} & Q_{2,n} \\ 0 & 0 & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & Q_{n-1,n} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}, \quad \widetilde{P} = \begin{pmatrix} \lambda_1^{(N-1)} & 0 \\ * & \Lambda^{(N-1)} \end{pmatrix}.$$

A set of subspaces $F'_k = \operatorname{im}\left(A^{(N-1)} - \lambda_1^{(N-1)}\mathrm{I}\right) \cap \operatorname{im}\left(A^{(N-1)} - \lambda_2^{(N-1)}\mathrm{I}\right) \cap \dots \cap \operatorname{im}\left(A^{(N-1)} - \lambda_k^{(N-1)}\mathrm{I}\right)$ form a complete flag too:

$$\mathbb{C}^n = F'_0 \supset F'_1 \supset \dots \supset F'_{n-1}.$$

The identity $Q = \mathrm{I}$ means the coinciding of the flag with the coordinate flag again. Let us denote

$$l_k := F_k \cap F'_{k-1}, \quad k = 1, 2, \dots, n; \quad F'_0 := \mathbb{C}^n.$$

If $A^{(N)}$ and $A^{(N-1)}$ are in a general position, $\dim l_k = 1, \forall k$ because $\dim F_k + \dim F'_{k-1} = n + 1$.

We get n (coordinate) directions in $\mathbb{C}^n$, but it is not enough for the fixation of the frame in the space. We must to coordinate the scales along the directions.

Let us fix one more direction, let it be an eigenspace $\ker\left(A^{(N-2)} - \lambda_1^{(N-2)}\mathrm{I}\right)$ of some $A^{(N-2)}$. We denote it by l_0:

$$l_0 := \ker\left(A^{(N-2)} - \lambda_1^{(N-2)}\mathrm{I}\right).$$

Stated above construction defines $n + 1$ directions in $\mathbb{C}^n$. *These directions uniquely fix a projective frame such that* $\mathbf{e}_k$ *is parallel to* l_k *and* $\sum_k \mathbf{e}_k$ *is parallel to* l_0. The construction can be treated as choosing such a section of the fiber bundle

$$\mathcal{O}^{(1)} \times \mathcal{O}^{(2)} \times \dots \times \mathcal{O}^{(N)} \to \mathcal{O}^{(1)} \times \mathcal{O}^{(2)} \times \dots \times \mathcal{O}^{(N)}/\mathrm{GL}(n, \mathbb{C})$$

that $A^{(N)}$ and $A^{(N-1)}$ are

$$\begin{pmatrix} \lambda_1^{(N)} & a_1^2 & a_1^3 & \dots & a_1^n \\ 0 & \lambda_2^{(N)} & a_2^3 & \dots & a_2^n \\ 0 & 0 & \lambda_3^{(N)} & \dots & a_3^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_n^{(N)} \end{pmatrix} \text{ and } \begin{pmatrix} \lambda_1^{(N-1)} & 0 & 0 & \dots & 0 \\ b_2^1 & \lambda_2^{(N-1)} & 0 & \dots & 0 \\ b_3^1 & b_3^2 & \lambda_3^{(N-1)} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_n^1 & b_n^2 & b_n^3 & \dots & \lambda_n^{(N-1)} \end{pmatrix}, \tag{2}$$

and $(1, 1, \dots, 1)$ is the eigenvector of $A^{(N-2)}$ corresponding $\lambda_1^{(N-2)}$.

4.2. Constant value of momentum map

We need to solve the equation

$$\sum_{k=1}^{N} A^{(k)} = 0.$$

It is N^2 scalar equations, but one of them has been solved automatically, it is $\sum_k \operatorname{tr} A^{(k)} = 0$, because eigenvalues of $A^{(k)}$'s are the parameters of the orbits. It must be satisfied in advance: $\sum_k \operatorname{tr} A^{(k)} = 0$.

Let us consider the first step of the iteration process of the construction of the flag coordinates for $A^{(N-2)}$. It gives

$$A^{(N-2)} = \begin{pmatrix} \lambda_1^{(N-2)} - c_\Sigma & \vec{c} \\ \left((\lambda - c_\Sigma)\mathrm{I} - \tilde{A}^{(N-2)}\right)\vec{1} & \tilde{A}^{(N-2)} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \vec{1}\cdot\vec{c} \end{pmatrix},$$

where

$$\vec{1}\cdot\vec{c} = \begin{pmatrix} c_2 & c & \dots & c_n \\ c_2 & c & \dots & c_n \\ \vdots & \vdots & \ddots & \vdots \\ c_2 & c & \dots & c_n \end{pmatrix} \quad \mathrm{gl}(n-1,\mathbb{C}), \quad \vec{c}\vec{1} =: c_\Sigma \quad \mathbb{C}.$$

Matrix $\tilde{A}^{(N-2)}$ is an arbitrary matrix from the orbit $\widetilde{\mathcal{O}}^{(N-2)}$. The orbit is defined by the eigenvalues $\lambda_2^{(N-2)}, \lambda^{(N-2)}, \dots, \lambda_n^{(N-2)}$.

Consider the diagonal part of $\sum_k A^{(k)} = 0$ first. The $n-1$ equations will be satisfied if we set

$$c_s := -\left(\widetilde{A}^{(N-2)} + \sum_{k=1}^{N-} A^{(k)}\right)_{ss} - \lambda_s^{(N)} - \lambda_s^{(N-1)}, \quad s = 2,3,\dots,n.$$

The vanishing of the last diagonal element follows from the equality $\sum_k \operatorname{tr} A^{(k)} = 0$.

The equations for non-diagonal elements of $\sum_k A^{(k)} = 0$ will be solved due to (2):

$$a_i^j := -\left(\sum_{k=1}^{N-2} A^{(k)}\right)_{ij}, \quad b_j^i := -\left(\sum_{k=1}^{N-2} A^{(k)}\right)_{ji}.$$

5. Example: Painlevé case

Let us consider the simplest non-trivial case that is four poles and 2×2 matrices. All the theory is invariant with respect to the Möbius transformation of the variable z, so we move three poles to $\{0,1,\infty\}$, and denote by t the position of the rest pole:

$$z_1 = 0, \quad z_2 = \infty, \quad z\ \ = t, \quad z_4 = 1.$$

The above-stated method of the parameterization of

$$\mathcal{O}^{(1)} \times \mathcal{O}^{(2)} \times \mathcal{O}^{(\ )} \times \mathcal{O}^{(4)} /\!\!/ \mathrm{GL}(2, \mathbb{C})$$

gives the following explicit form of the Fuchsian system

$$\frac{d}{dz}\Psi = \left(\frac{A^{(1)}}{z} + \frac{A^{(\ )}}{z-t} + \frac{A^{(4)}}{z-1} \right)\Psi, \quad A^{(2)} = -\sum_{k} A^{(k)},$$

$$A^{(1)} = \begin{pmatrix} \lambda_1 - pq & p \\ -q(pq - 2\lambda_1) & pq - \lambda_1 \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} pq - \lambda_\Sigma + \lambda_2 & \lambda_\Sigma - pq \\ pq - \lambda_\Sigma + 2\lambda_2 & \lambda_\Sigma - \lambda_2 - pq \end{pmatrix},$$
$$A^{(3)} = \begin{pmatrix} \lambda_3 & p(q-1) - \lambda_\Sigma \\ 0 & -\lambda_3 \end{pmatrix}, \quad A^{(4)} = \begin{pmatrix} \lambda_4 & 0 \\ q(p(q-1) - 2\lambda_1) + \lambda_\Sigma - 2\lambda_2 & -\lambda_4 \end{pmatrix}.$$

The eigenvalues of $A^{(k)}$ are denoted by $\pm\lambda_k$ and $\sum_{k=1}^{4} \lambda_k =: \lambda_\Sigma$. The numeration of the matrices was fixed in the previous sections, and the reason of the present numeration of the poles z_k is the simplification of the calculation of the Hamiltonian. We move to infinity the most bulky matrix-residue and take as t the pole with the simplest one. The calculation of the Hamiltonian corresponding the dynamics with respect to $z\ \ = t$ gives:

$$\begin{aligned} H &= \sum_{k \neq} \operatorname{tr} A^{(\ )} A^{(k)} \frac{dz\ \ - dz_k}{z\ \ - z_k} \Bigg|_{dz_k = 0} \Big/ dz \\ &= \frac{1}{t(t-1)} \operatorname{tr} A^{(\ )} \left((t-1)A^{(1)} + tA^{(4)} \right) \\ &= \frac{1}{t(t-1)} \left(q(q-1)(q-t) \left(p^2 - p\left(\frac{c_1}{q} + \frac{c}{q-t} + \frac{c_4}{q-1} \right) \right) + c_2 q \right) + \mathrm{const}_t, \end{aligned}$$

where $c_1 = \lambda_1 - \lambda_2 + \lambda\ \ + \lambda_4$, $c_2 = 2\lambda_1(\lambda_1 + \lambda_2 + \lambda\ \ + \lambda_4)$, $c\ \ = \lambda_1 + \lambda_2 + \lambda\ \ - \lambda_4$, $c_4 = \lambda_1 + \lambda_2 - \lambda\ \ + \lambda_4$ are the parameters of the Isomonodromic Deformation, and const_t does not depend on p, q. It is well known Hamiltonian of Painlevé VI system, its Euler–Lagrange equation is Painlevé VI equation.

The resulting formulae are valid for *any* values of the eigenvalues $A^{(k)}$ $\mathrm{sl}(2, \mathbb{C})$. We have two orbits if the eigenvalues $\pm\lambda_k$ coincide (i.e. vanish). One of them consists of Jordan matrices and the second one is just zero matrix. The formulae describe the case of the maximal dimension that is the orbits of Jordan boxes.

Note that the presented parameterization of the phase space is not unique. We can permute the poles z_k, it gives the action of the Σ_4-symmetry group on the Painlevé VI equation. The so-called Okamoto-symmetry corresponds to the change of the scales on the coordinate lines, namely instead of the fixation of the direction of the eigenvector of $A^{(2)}$ we can fix the value of its non-diagonal matrix element.

6. Conclusion

The phase space of the Isomonodromic Deformation equations is

$$\mathcal{O}^{(1)} \times \mathcal{O}^{(2)} \times \cdots \times \mathcal{O}^{(N)} /\!\!/ \mathrm{GL}(n, \mathbb{C}).$$

The canonical parameters on it are the flag coordinates on $N-3$ orbits $\mathcal{O}^{(k)}, k = 1, \ldots, N-3$, and the coordinates constructed on $\mathcal{O}^{(N-2)}$, except those constructed on the first step of the iteration process.

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On some deformations of the Poisson structure associated with the algebroid bracket of differential forms

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Abstract. We use the algebroid bracket of differential forms to generate the Poisson structure π_C on the tangent bundle TM. Next, we present how to construct deformations of this structure starting from the initial Poisson structure π on a manifold M.

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1. Introduction

In this paper we shall first recall some basic notions from Poisson geometry. We consider the concept of lifting multivectors from manifold M to TM. The complete lift of Poisson tensor π from Poisson manifold M is the same as the Poisson structure π_C associated with the algebroid bracket of differential forms on T^*M. Next for a certain class of tensors on M we show how to build deformations of the Poisson structure π_C. In the end, we will explain the relationship of this construction with the geometry of these objects. We also present some examples of application to the theory of real low-dimensional Lie algebras.

2. Lifting of Poisson structure

Let (M, π) be a N-dimensional Poisson manifold. Then the Poisson tensor π $\Gamma\left(\bigwedge^2 TM\right)$ in a system of local coordinates $\mathbf{x} = (x_1, \ldots, x_N)$ on M can be written as

$$\pi(\mathbf{x}) = \sum_{i,j=1}^{N} \frac{1}{2}\pi_{ij}(\mathbf{x})\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \tag{1}$$

where $\pi_{ij}(\mathbf{x}) = -\pi_{ji}(\mathbf{x}) = \{x_i, x_j\}$ satisfies the following system of equations equivalent to the Jacobi identity

$$\sum_{s=1}^{N} \left(\frac{\partial \pi_{ij}}{\partial x_s}\pi_{sk} + \frac{\partial \pi_{ki}}{\partial x_s}\pi_{sj} + \frac{\partial \pi_{jk}}{\partial x_s}\pi_{si} \right) = 0. \tag{2}$$

The Poisson bracket on M is given by $\{f, g\} = \pi(df, dg)$ and it is a skew-symmetric bilinear mapping satisfying the Jacobi identity and the Leibniz rule.

We say that manifold M with two Poisson tensors π_1, π_2 is a bi-Hamiltonian manifold if they are compatible, i.e., their linear combination is again a Poisson tensor $\pi_\lambda = \pi_1 + \lambda\pi_2$.

Next we recall the concept of lifting multivectors from M to TM, see [6,9,13]. If we have a multivector field on M (in local coordinates) given as

$$X = \sum_{i_1,\ldots,i_k=1}^{N} v_{i_1\ldots i_k}(\mathbf{x})\frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_k}} \tag{3}$$

then the complete lift to TM is given by formula

$$\begin{aligned} X_C = \sum_{i_1,\ldots,i_k=1}^{N} \Bigg(& v_{i_1\ldots i_k}(\mathbf{x})\frac{\partial}{\partial y_{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial y_{i_{l-1}}} \wedge \frac{\partial}{\partial x_{i_l}} \wedge \frac{\partial}{\partial y_{i_{l+1}}} \wedge \cdots \wedge \frac{\partial}{\partial y_{i_k}} \\ & + \sum_{s=1}^{N} \frac{\partial v_{i_1\ldots i_k}}{\partial x_s}(\mathbf{x})y_s\frac{\partial}{\partial y_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial y_{i_k}} \Bigg), \end{aligned} \tag{4}$$

where $(\mathbf{x}, \mathbf{y}) = (x_1, \ldots, x_N, y_1, \ldots, y_N)$ is a system of local coordinates on TM induced by the coordinates $(x_1, \ldots, x_N)$ on M. Similarly the vertical lift is defined by

$$X_V = \sum_{i_1,\ldots,i_k=1}^{N} v_{i_1\ldots i_k}(\mathbf{x})\frac{\partial}{\partial y_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial y_{i_k}}. \tag{5}$$

If we apply it to the Poisson tensor (1), we get

$$\begin{aligned} \pi_C(\mathbf{x}, \mathbf{y}) = \sum_{1\le i<j}^{N} \Bigg(& \pi_{ij}(\mathbf{x})\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j} + \pi_{ij}(\mathbf{x})\frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial x_j} \\ & + \sum_{s=1}^{N} \frac{\partial \pi_{ij}}{\partial x_s}(\mathbf{x})y_s\frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j} \Bigg). \end{aligned} \tag{6}$$

Now, we consider a certain Lie algebroid structure on T^*M. The algebroid bracket of differential forms on one-forms is given by the following formula:

$$[df, dg] = d\{f, g\} \tag{7}$$

where $f, g \quad C^{\infty}(M)$. This bracket also satisfies the following conditions:

$$[df, h\,dg] = h[df, dg] + a(df)(h)dg, \tag{8}$$

$$a\left([df, dg]\right) = [a(df), a(dg)], \tag{9}$$

for all $df, dg \quad \Gamma(T^*M)$, $h \quad C^{\infty}(M)$, see [7, 8], where anchor $a : T^*M \to TM$ is defined by Poisson bracket as follows $a(df)(\cdot) = \{f, \cdot\}$. On the dual space TM to the Lie algebroid T^*M we have the tangent Poisson structure

$$\{f \circ q_M, g \circ q_M\}_{TM} = 0, \tag{10}$$

$$\{l_{df}, l_{dg}\}_{TM} = l_{[df,dg]}, \tag{11}$$

$$\{f \circ q_M, l_{dg}\}_{TM} = -a(dg)(f) \circ q_M, \tag{12}$$

where $f, g \quad C^{\infty}(M)$ and $q_M : TM \to M$. In the above formulas $l_{df} \quad C^{\infty}(TM)$ is defined by pairing $l_{df}(X) = \langle X, df(q_M(X))\rangle$, where $X \quad TM$. So the Poisson structure is related to the structure (6) and can be represented by a matrix

$$\pi_C(\mathbf{x}, \mathbf{y}) = \left(\begin{array}{c|c} 0 & \pi(\mathbf{x}) \\ \hline \pi(\mathbf{x}) & \sum_{s=1}^{N} \frac{\partial \pi}{\partial x_s}(\mathbf{x}) y_s \end{array}\right), \tag{13}$$

where $y_s = l_{dx_s}$.

3. Some deformations of the Poisson structure

In this section, we present some deformations of the Poisson structure (13) on TM and we give a geometrical interpretation of them. In addition, we also apply them to the example based on Lie algebra $\mathfrak{e}(2)$.

We consider a one-parameter deformation of a Poisson structure π_C on TM given by

$$\pi_\lambda = \pi_C + \lambda \tilde{\pi}. \tag{14}$$

It is possible to describe some of them starting from the initial Poisson structure π on the manifold M.

The particular case of above construction, when a Poisson structure π on the manifold M satisfies a specific condition, is described by the following theorem.

Proposition 1. *Let (M, π) be a Poisson manifold and let $\mathbf{x} = (x_1, \dots, x_N)$ be a system of local coordinates on M. If the Poisson tensor π does not depend on the variable x_p for certain $1 \leq p \leq N$, and the function c is Casimir function for π, then we obtain a new Poisson tensor on TM*

$$\pi_{C,c(\mathbf{x})}(\mathbf{x}, \mathbf{y}) = \left(\begin{array}{c|c} 0 & \pi(\mathbf{x}) + E_{pp}c(\mathbf{x}) \\ \hline \pi(\mathbf{x}) - E_{pp}c(\mathbf{x}) & \sum_{s=1}^{N} \frac{\partial \pi}{\partial x_s}(\mathbf{x}) y_s \end{array}\right), \tag{15}$$

where $(\mathbf{x}, \mathbf{y}) = (x_1, \dots, x_N, y_1, \dots, y_N)$ is a system of local coordinates on TM and E_{pp} is the $N \times N$ matrix all of whose entries are zero except the diagonal (p, p) entry which is one.

Proof. If $c(\mathbf{x}) = 0$, then we obtain the classical tangent Poisson structure (13). For $c(\mathbf{x}) \neq 0$, by a direct calculation we obtain

$$
\begin{aligned}
&\circlearrowleft \{\{x_i, x_j\}_{C,c(\mathbf{x})}, x_k\}_{C,c(\mathbf{x})} = \circlearrowleft \{\{x_i, x_j\}_{C,c(\mathbf{x})}, y_k\}_{C,c(\mathbf{x})} = 0, \\
&\circlearrowleft \{\{y_i, y_j\}_{C,c(\mathbf{x})}, x_k\}_{C,c(\mathbf{x})} \\
&\quad = \sum_{m=1}^{N} \delta_{pk} \left(\delta_{pi}\pi_{mj}(\mathbf{x}) - \delta_{pj}\pi_{mi}(\mathbf{x})\right) \frac{\partial c}{\partial x_m}(\mathbf{x}) \\
&\quad\quad + c(\mathbf{x}) \left(\delta_{pi}\frac{\partial \pi_{jk}}{\partial x_p}(\mathbf{x}) + \delta_{pj}\frac{\partial \pi_{ki}}{\partial x_p}(\mathbf{x}) - \delta_{pk}\frac{\partial \pi_{ij}}{\partial x_p}(\mathbf{x})\right) = 0, \\
&\circlearrowleft \{\{y_i, y_j\}_{C,c(\mathbf{x})}, y_k\}_{C,c(\mathbf{x})} \\
&\quad = \sum_{s=1}^{N} y_s c(\mathbf{x}) \left(\delta_{pi}\frac{\partial^2 \pi_{jk}}{\partial x_p \partial x_s}(\mathbf{x}) + \delta_{pj}\frac{\partial^2 \pi_{ki}}{\partial x_p \partial x_s}(\mathbf{x}) + \delta_{pk}\frac{\partial^2 \pi_{ij}}{\partial x_p \partial x_s}(\mathbf{x})\right),
\end{aligned}
\tag{16}
$$

because c is Casimir function for π and π does not depend on the variable x_p. Here $\circlearrowleft \{\{f, g\}, h\}$ indicates the sum over circular permutations of f, g, h. This finishes the proof. □

Remark. If $M = \mathbb{R}^n$ is a Poisson manifold, where the Poisson tensor π does not depend on the variable x_p and the function c is a linear Casimir function for π, then

$$
\pi_{C,c(\mathbf{y})}(\mathbf{x}, \mathbf{y}) = \left(\begin{array}{c|c} 0 & \pi(\mathbf{x}) + E_{pp}c(\mathbf{y}) \\ \hline \pi(\mathbf{x}) - E_{pp}c(\mathbf{y}) & \sum_{s=1}^{N} \frac{\partial \pi}{\partial x_s}(\mathbf{x}) y_s \end{array}\right), \tag{17}
$$

is also a Poisson tensor on TM.

Proposition 2. *If $c_1, \dots, c$, where $r = \dim M - \operatorname{rank}\pi$, are Casimir functions for the Poisson structure π and if all c_i do not depend on the variable x_p for certain $1 \leq p \leq N$, then the functions*

$$
c_i \circ q_M \quad \textit{and} \quad l_{dc_i} = \sum_{s=1}^{N} \frac{\partial c_i}{\partial x_s}(\mathbf{x}) y_s, \quad i = 1, \dots r, \tag{18}
$$

are Casimir functions for the Poisson tensor $\pi_{C,c(\mathbf{x})}$ and $\pi_{C,c(\mathbf{y})}$.

Proof. Let us take the Poisson tensor $\pi_{C,c(\mathbf{x})}$. A direct calculation gives us

$$\begin{aligned}
\{c_i(\mathbf{x}), x_j\}_{C,c(\mathbf{x})} &= 0,\\
\{c_i(\mathbf{x}), y_j\}_{C,c(\mathbf{x})} &= \sum_{s=1}^{N} \frac{\partial c_i}{\partial x_s}(\mathbf{x})\pi_{sj}(\mathbf{x}) + \delta_{pj}c(\mathbf{x})\frac{\partial c_i}{\partial x_p}(\mathbf{x}) = 0,\\
\{l_{dc_i}, x_j\}_{C,c(\mathbf{x})} &= \sum_{s=1}^{N} \frac{\partial c_i}{\partial x_s}(\mathbf{x})\pi_{sj}(\mathbf{x}) - \delta_{pj}c(\mathbf{x})\frac{\partial c_i}{\partial x_p}(\mathbf{x}) = 0,\\
\{l_{dc_i}, y_j\}_{C,c(\mathbf{x})} &= \sum_{s,m=1}^{N} y_m \frac{\partial}{\partial x_m}\left(\pi_{sj}(\mathbf{x})\frac{\partial c_i}{\partial x_s}(\mathbf{x})\right) + \delta_{pj}c(\mathbf{x})\frac{\partial^2 c_i}{\partial x_p \partial x_s}(\mathbf{x})y_s = 0,
\end{aligned}$$

because $c_i(\mathbf{x})$ is a Casimir function for π and $c_i(\mathbf{x})$ does not depend on the variable x_p. The proof for the structure $\pi_{C,c(\mathbf{y})}$ is completely analogous. □

Example 1. Let us consider the Euclidean Lie algebra $\mathfrak{e}(2)$. The commutation rules for $\mathfrak{e}(2)$ are $[e_1, e\ ] = -e_2$, $[e_2, e\ ] = e_1$. On $\mathbb{R}$ with coordinates $(x_1, x_2, x\ )$ we have the linear Poisson structure

$$\pi(x_1, x_2, x\ ) = \begin{pmatrix} 0 & 0 & -x_2 \\ 0 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{pmatrix} \tag{19}$$

associated with this Lie algebra. This tensor has one Casimir $c = x_1^2 + x_2^2$. Then on $T\mathbb{R}$ we have the Poisson tensor

$$\pi_C(x_1, x_2, x\ , y_1, y_2, y\ ) = \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & -x_2 \\ 0 & 0 & 0 & 0 & 0 & x_1 \\ 0 & 0 & 0 & x_2 & -x_1 & 0 \\ \hline 0 & 0 & -x_2 & 0 & 0 & -y_2 \\ 0 & 0 & x_1 & 0 & 0 & y_1 \\ x_2 & -x_1 & 0 & y_2 & -y_1 & 0 \end{array}\right). \tag{20}$$

Using the constructions described in Proposition 1 we obtain the following deformation of Poisson structure on $T\mathbb{R}$:

$$\pi_{C,c(\mathbf{x})}(\mathbf{x}, \mathbf{y}) = \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & -x_2 \\ 0 & 0 & 0 & 0 & 0 & x_1 \\ 0 & 0 & 0 & x_2 & -x_1 & x_1^2 + x_2^2 \\ \hline 0 & 0 & -x_2 & 0 & 0 & -y_2 \\ 0 & 0 & x_1 & 0 & 0 & y_1 \\ x_2 & -x_1 & x_1^2 + x_2^2 & y_2 & -y_1 & 0 \end{array}\right). \tag{21}$$

In this case, the Casimir functions (Proposition 2) are given by the formulas

$$c(\mathbf{x}, \mathbf{y}) = x_1^2 + x_2^2, \quad l_{dc}(\mathbf{x}, \mathbf{y}) = 2x_1 y_1 + 2x_2 y_2. \tag{22}$$

Proposition 1 also has a geometric interpretation. But at the beginning we recall the well-known notion of a Poisson vector field. A smooth vector field X on a Poisson manifold M such that $\mathcal{L}_X \pi = 0$ or equivalently

$$X(\{f, g\}) = \{X(f), g\} + \{f, X(g)\} \tag{23}$$

is called a Poisson vector field. Next, we observe that if a Poisson tensor π does not contain the variable x_p, then $X = \frac{\partial}{\partial x_p}$ is a Poisson vector field for its and also for a Poisson tensor π_C. In addition, $Y = \frac{\partial}{\partial y_p}$ is also a vector field for a Poisson tensor π_C. More generally if we find Poisson vector fields X, Y for a Poisson tensor π on M, which commute $[X, Y] = 0$, we can construct deformations at the level of manifold TM. Using complete X_C and vertical Y_V lifts of these Poisson vector fields, we can build a Poisson tensor $\tilde{\pi} = X_C \wedge Y_V$ that is compatible with the initial Poisson tensor π_C. The manifold $(TM, \pi_C, \tilde{\pi})$ equipped with these Poisson structures is a bi-Hamiltonian manifold. This of course means that $X_C \wedge Y_V$ belongs to second Poisson cohomology group $H^2_{\pi_C}(TM)$ and can be used as infinitesimal deformation of π_C, see [3, 5].

Example 2. If we again take the Euclidean Lie algebra $\mathfrak{e}(2)$ then we find that the Poisson vector field with the first degree polynomial coefficients has the form

$$X = (ax_1 + bx_2)\frac{\partial}{\partial x_1} + (-bx_1 + ax_2)\frac{\partial}{\partial x_2} + (cx_1 + dx_2 + e)\frac{\partial}{\partial x}, \tag{24}$$

where $a, b, c, d, e \quad \mathbb{R}$.

1. Let us take now Poisson vector fields

$$X = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}, \quad Y = \frac{\partial}{\partial x}. \tag{25}$$

   Then their complete and vertical lifts are of the form

$$X_C = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} + y_2 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial y_2}, \quad Y_V = \frac{\partial}{\partial y}. \tag{26}$$

   They form the bi-vector

$$\begin{aligned} X_C \wedge Y_V = {} & x_2 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y} - x_1 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial y} \\ & + y_2 \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial y} - y_1 \frac{\partial}{\partial y_2} \wedge \frac{\partial}{\partial y}, \end{aligned} \tag{27}$$

   which can be considered as a deformation of the Poisson tensor (20), i.e., we get a linear Poisson structure on $T\mathbb{R}$

$$\pi_C + X_C \wedge Y_V = x_2 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y_1} - x_1 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y_2}. \tag{28}$$

We recognize the Lie–Poisson structure related to direct sum $\mathcal{A}_{5,1} \oplus \langle y\ \rangle$. An isomorphism is given by mapping

$$(x_1, x_2, x\ , y_1, y_2, y\ ) \mapsto (e_1, -e_2, e_5, e_4, e\ , e_6).$$

The commutation rules for Lie algebra $\mathcal{A}_{5,1}$ are $[e\ , e_5] = e_1$, $[e_4, e_5] = e_2$, see [10]. Because the Lie derivatives for the Casimir function $c = x_1^2 + x_2^2$ for the Poisson tensor (19) along X and Y vanish, i.e., $\mathcal{L}_X c = 0 = \mathcal{L}_Y c$, then c and l_{dc} are Casimir functions for (28). However x_1 and x_2 are also Casimir functions. Finally, we have three Casimir functions x_1, x_2, l_{dc} for $\mathcal{A}_{5,1}$.

2. Let us take now Poisson vector fields

$$X = (x_1 + x_2)\frac{\partial}{\partial x_1} + (-x_1 + x_2)\frac{\partial}{\partial x_2}, \quad Y = \frac{\partial}{\partial x}. \tag{29}$$

Then their complete and vertical lifts are of the form

$$\begin{aligned} X_C =& (x_1 + x_2)\frac{\partial}{\partial x_1} + (-x_1 + x_2)\frac{\partial}{\partial x_2} \\ &+ (y_1 + y_2)\frac{\partial}{\partial y_1} + (-y_1 + y_2)\frac{\partial}{\partial y_2}, \\ Y_V =& \frac{\partial}{\partial y}. \end{aligned} \tag{30}$$

They form the bi-vector $X_C \wedge Y_V$ which can be considered as a deformation of the Poisson tensor (20), i.e., we get a linear Poisson structure on $T\mathbb{R}$

$$\begin{aligned} \pi_C + X_C \wedge Y_V = x_1 &\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y} + x_2 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial y} \\ &+ x_2 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y_1} - x_1 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y_2} \\ &+ y_1 \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial y} + y_2 \frac{\partial}{\partial y_2} \wedge \frac{\partial}{\partial y}. \end{aligned} \tag{31}$$

We recognize the Lie–Poisson structure related to Lie algebra $\mathcal{S}_{6,140}$. An isomorphism is given by mapping

$$(x_1, x_2, x\ , y_1, y_2, y\ ) \mapsto (e_1, -e_2, e_5, e_4, e\ , -e_6).$$

The commutation rules for Lie algebra $\mathcal{S}_{6,140}$ are $[e\ , e_5] = e_1$, $[e_4, e_5] = e_2$, $[e_6, e_1] = e_1$, $[e_6, e_2] = e_2$, $[e_6, e\ ] = e\ $, $[e_6, e_4] = e_4$, see [11]. The Casimir functions have the form

$$\frac{x_2}{x_1} \quad \text{and} \quad \frac{x_1 y_1 + x_2 y_2}{x_1 x_2}.$$

In addition, based on work [2, 4], we can make subsequent modifications to the Poisson tensor π_C.

Proposition 3. *Let (M, π_1, π_2) be a bi-Hamiltonian manifold and the Poisson tensors π_1 and π_2 do not depend on the variable x_p for a certain $1 \leq p \leq N$.*

1. *If the function c is a Casimir function for π_2, then the structure on TM*

$$\pi_{C,\lambda,c(\mathbf{x})}(\mathbf{x},\mathbf{y}) = \left(\begin{array}{c|c} 0 & \pi_2(\mathbf{x}) + E_{pp}c(x) \\ \hline \pi_2(\mathbf{x}) - E_{pp}c(x) & \sum_{s=1}^{N} \frac{\partial \pi_2}{\partial x_s}(\mathbf{x})y_s + \lambda\pi_1(\mathbf{x}) \end{array}\right) \tag{32}$$

   is a Poisson tensor.
2. *If $M = \mathbb{R}^n$ and the function c is a linear Casimir function for π_2, then the structure on TM*

$$\pi_{C,\lambda,c(\mathbf{y})}(\mathbf{x},\mathbf{y}) = \left(\begin{array}{c|c} 0 & \pi_2(\mathbf{x}) + E_{pp}c(y) \\ \hline \pi_2(\mathbf{x}) - E_{pp}c(y) & \sum_{s=1}^{N} \frac{\partial \pi_2}{\partial x_s}(\mathbf{x})y_s + \lambda\pi_1(\mathbf{x}) \end{array}\right) \tag{33}$$

   is a Poisson tensor.

Proof. By direct calculation. □

Proposition 4. *Let $c_1, \ldots, c$, where $r = \dim M - \operatorname{rank} \pi_2$, be Casimir functions for the Poisson structure π_2 and functions f_i^λ, $i = 1, \ldots, r$, satisfy the conditions*

$$\{f_i^\lambda, x_j\}_2 = \{x_j, c_i\}_1, \quad \textit{for} \quad j = 1, \cdots, n. \tag{34}$$

If the functions c_i and f_i do not depend on the variable x_p for certain $1 \leq p \leq N$, then the functions

$$c_i \circ q_M \quad \textit{and} \quad \tilde{c}_i(\mathbf{x},\mathbf{y}) = \sum_{s=1}^{N} \frac{\partial c_i}{\partial x_s}(\mathbf{x})y_s + \lambda f_i^\lambda(\mathbf{x}), \quad i = 1, \ldots, r, \tag{35}$$

are Casimir functions for the Poisson tensor $\pi_{C,\lambda,c(\mathbf{x})}$ and $\pi_{C,\lambda,c(\mathbf{y})}$ given by (32), (33)*, respectively.*

Proof. The proof is analogous to the proof of Proposition 2. □

Some results were investigated also in the works [1, 12] for some similar constructions.

In the case of a linear Poisson structure, when $M = \mathfrak{g}^*$ is the dual to Lie algebra $\mathfrak{g}$, we have additional Poisson structures on TM.

Proposition 5. *Let π be the Lie–Poisson structure on $\mathfrak{g}^*$, which does not depend on the variable x_p.*

1. *If c is a Casimir function for π then the tensor*

$$\widetilde{\pi}_{T\mathfrak{g}^*,c(\mathbf{x})}(\mathbf{x},\mathbf{y}) = \left(\begin{array}{c|c} \lambda\pi(\mathbf{x}) & \pi(\mathbf{x}) + E_{pp}c(x) \\ \hline \pi(\mathbf{x}) - E_{pp}c(x) & \pi(\mathbf{y}) \end{array}\right) \tag{36}$$

   gives a Poisson structure on $T\mathfrak{g}^*$ for any λ $\mathbb{R}$.

2. If c is a linear Casimir function for π then the tensors

$$\widetilde{\pi}_{T\mathfrak{g}^*,c(\mathbf{y})}(\mathbf{x},\mathbf{y}) = \left(\begin{array}{c|c} \lambda\pi(\mathbf{x}) & \pi(\mathbf{x}) + E_{pp}c(y) \\ \hline \pi(\mathbf{x}) - E_{pp}c(y) & \pi(\mathbf{y}) \end{array}\right), \tag{37}$$

$$\widetilde{\widetilde{\pi}}_{T\mathfrak{g}^*,c(\mathbf{x})}(\mathbf{x},\mathbf{y}) = \left(\begin{array}{c|c} \lambda\pi(\mathbf{y}) & \pi(\mathbf{x}) + E_{pp}c(x) \\ \hline \pi(\mathbf{x}) - E_{pp}c(x) & \pi(\mathbf{y}) \end{array}\right), \tag{38}$$

$$\widetilde{\widetilde{\pi}}_{T\mathfrak{g}^*,c(\mathbf{y})}(\mathbf{x},\mathbf{y}) = \left(\begin{array}{c|c} \lambda\pi(\mathbf{y}) & \pi(\mathbf{x}) + E_{pp}c(y) \\ \hline \pi(\mathbf{x}) - E_{pp}c(y) & \pi(\mathbf{y}) \end{array}\right) \tag{39}$$

give Poisson structures on $T\mathfrak{g}^*$ for any λ $\mathbb{R}$.

Proof. By direct calculation. ☐

Proposition 6. *Let $c_1, \dots, c$, where $r = \dim M - \operatorname{rank}\pi$, be Casimir functions for the Poisson structure π and let c_i do not depend on the variable x_p for certain $1 \le p \le N$.*

- *The functions*

$$c_i(\mathbf{x}) \quad \text{and} \quad \tilde{\tilde{c}}_i = c_i(\mathbf{x} - \lambda\mathbf{y}) - c_i(\mathbf{x}), \quad i = 1, \dots r, \tag{40}$$

are Casimir functions for the Poisson tensor $\widetilde{\pi}_{T\mathfrak{g}^*,c(\mathbf{x})}$ and $\widetilde{\pi}_{T\mathfrak{g}^*,c(\mathbf{y})}$ given by (36) and (37).

- *The functions*

$$\hat{c}_i(\mathbf{x},\mathbf{y}) = c_i(\mathbf{x} - \sqrt{\lambda}\mathbf{y}) + c_i(\mathbf{x} + \sqrt{\lambda}\mathbf{y}) \tag{41}$$

and

$$\hat{\hat{c}}_i = c_i(\mathbf{x} - \sqrt{\lambda}\mathbf{y}) - c_i(\mathbf{x} + \sqrt{\lambda}\mathbf{y}), \quad i = 1, \dots r, \tag{42}$$

are Casimir functions for the Poisson tensor $\widetilde{\widetilde{\pi}}_{T\mathfrak{g}^*,c(\mathbf{x})}$ and $\widetilde{\widetilde{\pi}}_{T\mathfrak{g}^*,c(\mathbf{y})}$ given by (38) and (39).

Proof. By direct calculation. ☐

Proposition 7. *Let $(\mathfrak{g}^*, \pi_1, \pi_2)$ be a bi-Hamiltonian manifold and the Poisson tensors π_1 and π_2 do not depend on the variable x_p for certain $1 \le p \le N$. If the function c is a Casimir function for π_2, then the structure on $T\mathfrak{g}$*

$$\widetilde{\pi}_{T\mathfrak{g},\lambda,c(\mathbf{x})}(\mathbf{x},\mathbf{y}) = \left(\begin{array}{c|c} \epsilon\pi_2(\mathbf{x}) & \pi_2(\mathbf{x}) + E_{pp}c(x) \\ \hline \pi_2(\mathbf{x}) - E_{pp}c(x) & \sum_{s=1}^{N} \frac{\partial \pi_2}{\partial x_s}(\mathbf{x}) y_s + \lambda\pi_1(\mathbf{x}) - \lambda\epsilon\pi_1(\mathbf{y}) \end{array}\right) \tag{43}$$

is Poisson tensor.

Proof. By direct calculation. ☐

Example 3. Starting from Lie algebra $\mathfrak{e}(2)$ and using Proposition 3, we obtain

$$\pi_{C,c(\mathbf{x})}(\mathbf{x},\mathbf{y}) = \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & -x_2 \\ 0 & 0 & 0 & 0 & 0 & x_1 \\ 0 & 0 & 0 & x_2 & -x_1 & x_1^2+x_2^2 \\ \hline 0 & 0 & -x_2 & 0 & 0 & -y_2+\lambda x_1 \\ 0 & 0 & x_1 & 0 & 0 & y_1-\lambda x_2 \\ x_2 & -x_1 & x_1^2+x_2^2 & y_2 & -y_1 & 0 \end{array}\right), \tag{44}$$

where we use the compatible Poisson structure related to Lie algebra $\mathcal{A}_{\ ,4}$. In this case, the Casimir functions (Proposition 4) are given by the formulas

$$c_1(\mathbf{x},\mathbf{y}) = x_1^2+x_2^2, \quad \tilde{c}_1(\mathbf{x},\mathbf{y}) = 2x_1y_1+2x_2y_2-2\lambda x_1x_2. \tag{45}$$

From Proposition 5 we have

$$\pi_{C,c(\mathbf{x})}(\mathbf{x},\mathbf{y}) = \left(\begin{array}{ccc|ccc} 0 & 0 & -\lambda x_2 & 0 & 0 & -x_2 \\ 0 & 0 & \lambda x_1 & 0 & 0 & x_1 \\ \lambda x_2 & \lambda x_1 & 0 & x_2 & -x_1 & x_1^2+x_2^2 \\ \hline 0 & 0 & -x_2 & 0 & 0 & -y_2 \\ 0 & 0 & x_1 & 0 & 0 & y_1 \\ x_2 & -x_1 & x_1^2+x_2^2 & y_2 & -y_1 & 0 \end{array}\right). \tag{46}$$

In this case, the Casimir functions (Proposition 6) are given by the formulas

$$c_1(\mathbf{x},\mathbf{y}) = x_1^2+x_2^2, \quad \tilde{\tilde{c}}_1(\mathbf{x},\mathbf{y}) = \lambda^2y_1^2+\lambda^2y_2^2-2\lambda x_1y_1-2\lambda x_2y_2. \tag{47}$$

4. Conclusions

We have shown that by starting from the three-dimensional Lie algebra we can build a six-dimensional Lie algebra. Next, using some deformations of the initial Poisson structure related to the algebroid bracket of differential forms, we get other Lie algebras of dimension six or five. Therefore it seems to be interesting to apply this formalism and some of its modifications to the theory of classification of real low-dimensional Lie algebras.

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Generation of Painlevé V transcendents

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Abstract. An algorithm for generating solutions to the Painlevé V equation (the Painlevé V transcendents) is presented. The first step is to look for general one-dimensional Schrödinger Hamiltonians ruled by third degree polynomial Heisenberg algebras, which have fourth order differential ladder operators. It is realized then that there is a key function that must satisfy the Painlevé V equation. Conversely, by identifying systems ruled by a third degree polynomial Heisenberg algebra, in particular their four extremal states, this key function can be built straightforwardly. The simplest Painlevé V transcendents will be generated through this algorithm.

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1. Introduction

Nowadays there is a growing interest in the analysis of nonlinear phenomena, looking for possible connections between certain physical systems and nonlinear differential equations [1]. In particular, a link between third degree polynomial Heisenberg algebras (PHAs) and Painlevé V (PV) equation has been found [2–8]. Through this connection a method for generating solutions to the PV equation (the PV transcendents) can be designed (see for example [7]). The key point to this approach is to identify systems ruled by such algebra, in particular their extremal states. It is known that for the radial oscillator it is possible to define second order ladder operators and to identify in a simple way the associated extremal states. In addition, some of its SUSY partners are ruled by fourth order ladder operators, thus supplying a lot of realizations of the third degree PHA and consequently plenty of PV transcendents.

In the next section we will introduce the kth order intertwining technique. In Section 3 we shall discuss in general the PHAs, while in Section 4 we will address those of third degree. It will also be established the link between third-degree PHA and Painlevé V equation. In Section 5, the SUSY partners of the radial oscillator shall be studied, the subfamily ruled by third-degree PHA is going to be identified, and the corresponding PV transcendents will be generated. Our conclusions shall be presented in Section 6.

2. kth order intertwining technique

Let H_0 and H_1 be two Schrödinger Hamiltonians intertwined to each other as follows:

$$H_1 A_1^+ = A_1^+ H_0, \quad H_0 A_1^- = A_1^- H_1, \tag{1}$$

$$H_j = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_j(x), \quad j = 0, 1, \tag{2}$$

$$A_1^\pm = \frac{1}{\sqrt{2}}\left[\mp\frac{\mathrm{d}}{\mathrm{d}x} + w_1(x,\epsilon_1)\right]. \tag{3}$$

Thus, the following equations must be fulfilled:

$$w_1'(x,\epsilon_1) + w_1^2(x,\epsilon_1) = 2[V_0(x) - \epsilon_1], \tag{4}$$

$$V_1(x) = V_0(x) - w_1'(x,\epsilon_1), \tag{5}$$

where ϵ_1 is called factorization energy. If the procedure is repeated several times, after the kth step we will get:

$$H_k A_k^+ = A_k^+ H_{k-1}, \quad H_{k-1} A_k^- = A_k^- H_k, \tag{6}$$

$$A_k^\pm = \frac{1}{\sqrt{2}}\left[\mp\frac{\mathrm{d}}{\mathrm{d}x} + w_k(x,\epsilon_k)\right], \tag{7}$$

and hence

$$w_k'(x,\epsilon_k) + w_k^2(x,\epsilon_k) = 2[V_{k-1}(x) - \epsilon_k], \tag{8}$$

$$V_k(x) = V_{k-1}(x) - w_k'(x,\epsilon_k) = V_0(x) - \sum_{j=1}^{k} w_j'(x,\epsilon_j). \tag{9}$$

As a consequence, the Hamiltonians H_0 and H_k are intertwined by kth order differential operators [9–11], namely,

$$H_k B_k^+ = B_k^+ H_0, \quad H_0 B_k^- = B_k^- H_k, \tag{10}$$

$$B_k^+ = A_k^+ \cdots A_1^+, \quad B_k^- = A_1^- \cdots A_k^-. \tag{11}$$

As can be seen from equation (9), the determination of the potential $V_k(x)$ depends of $V_0(x)$ and the sequence of solutions $w_j(x,\epsilon_j), j = 1,\ldots,k$, which turn out to be determined by k solutions $w_1(x,\epsilon_j), j = 1,\ldots,k$ of the initial Riccati equation (4) for the k factorization energies involved [12].

3. Polynomial Heisenberg algebras

The polynomial Heisenberg algebras (PHAs) of degree $m-1$ are deformations of the Heisenberg–Weyl algebra [4], also with three generators $\{H, \mathcal{L}_m^+, \mathcal{L}_m^-\}$, such that:

$$[H, \mathcal{L}_m^\pm] = \pm\mathcal{L}_m^\pm, \tag{12}$$

$$[\mathcal{L}_m^-, \mathcal{L}_m^+] \equiv N_m(H+1) - N_m(H) \equiv P_{m-1}(H), \tag{13}$$

$$N_m(H) \equiv \mathcal{L}_m^+\mathcal{L}_m^- = \prod_{i=1}^{m} (H - \mathcal{E}_i)\,. \tag{14}$$

It is important to represent such PHA in its differential form, in which

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + V(x), \tag{15}$$

while $\mathcal{L}_m^\pm$ are mth order differential ladder operators. The spectrum of the Hamiltonian Sp(H) will depend of its extremal states, which satisfy $\mathcal{L}_m^-\psi_{\mathcal{E}_i} = 0$ and $H\psi_{\mathcal{E}_i} = \mathcal{E}_i\psi_{\mathcal{E}_i}$. We observe two possibilities:

(a) If $\{\psi_{\mathcal{E}_i}, i = 1, \dots, s\}$ satisfy the boundary conditions, where $s \le m$, then Sp(H) in general will contain s infinite energy ladders.

(b) If among these s extremal states the jth one is such that $(\mathcal{L}_m^+)^{n-1}\psi_{\mathcal{E}_j} \neq 0$ but $(\mathcal{L}_m^+)^n\psi_{\mathcal{E}_j} = 0$, then Sp($H$) will contain $s-1$ infinite ladders plus the jth one, which will be finite.

Next, let us explore systems ruled by a third degree PHA, that have fourth order differential ladder operators and are linked with the Painlevé V equation.

4. Third degree PHA

Let $\mathcal{L}_4^\pm$ be fourth-order ladder operators such that [7]

$$\mathcal{L}_4^+ = A_4^+A^+A_2^+A_1^+, \quad \mathcal{L}_4^- = A_1^-A_2^-A^-A_4^-, \tag{16}$$

$$H_{j+1}A_j^+ = A_j^+H_j, \quad H_jA_j^- = A_j^-H_{j+1}, \quad j = 1,2,3,\ \ , \tag{17}$$

$$H_5 = H_1 - 1 \equiv H - 1, \tag{18}$$

where

$$A_j^\pm = \frac{1}{\sqrt{2}}\left(\pm\frac{\mathrm{d}}{\mathrm{d}x} - f_j\right), \quad j = 1,2,3,\ \ . \tag{19}$$

Equations (16)–(19) lead to a system of coupled differential equations for the involved functions and potentials. When decoupling such system, the key at the end is to find a solution $w(z)$ to the Painlevé V (PV) equation:

$$w'' = \left(\frac{1}{2w} + \frac{1}{w-1}\right)w'^2 - \frac{w'}{z} + \frac{(w-1)^2}{z^2}\left(aw + \frac{b}{w}\right) + c\frac{w}{z} + d\frac{w(w+1)}{w-1}, \tag{20}$$

where $z = x^2$, the $'$ means here derivative with respect to z, and the parameters of the PV equation are related with the factorization energies as follows:

$$a = \frac{(\mathcal{E}_1 - \mathcal{E}_2)^2}{2}, \qquad b = -\frac{(\mathcal{E}\ \ - \mathcal{E}_4)^2}{2}, \tag{21}$$

$$c = \frac{\mathcal{E}_1 + \mathcal{E}_2 - \mathcal{E}\ \ - \mathcal{E}_4 - 1}{2}, \qquad d = -\frac{1}{8}. \tag{22}$$

The key functions of this treatment are related with $w(z)$ in the way:

$$g(x) = -f_1 - f_2 = \frac{x}{w(x^2) - 1}, \tag{23}$$

$$h(x) = -x - g(x) = -x - \frac{x}{w(x^2) - 1}. \tag{24}$$

The analogue of the number operator is a polynomial of degree 4 in H:

$$N_4(H) = \mathcal{L}_4^+ \mathcal{L}_4^- = (H - \mathcal{E}_1)(H - \mathcal{E}_2)(H - \mathcal{E}\ \ )(H - \mathcal{E}_4), \tag{25}$$

whose roots $\mathcal{E}_i$ have associated extremal states given by

$$\begin{aligned} \psi_{\mathcal{E}_1} \propto & \left[\frac{h}{2}\left(\frac{g'}{2g} - \frac{h'}{2h} - \frac{x}{2} + \frac{\mathcal{E}_2 - \mathcal{E}_1}{g}\right) - \mathcal{E}_1 + \frac{\mathcal{E}\ \ + \mathcal{E}_4}{2}\right] \\ & \times \exp\left[\int \left(\frac{g'}{2g} + \frac{g}{2} + \frac{\mathcal{E}_2 - \mathcal{E}_1}{g}\right) \mathrm{d}x\right], \end{aligned} \tag{26}$$

$$\begin{aligned} \psi_{\mathcal{E}_2} \propto & \left[\frac{h}{2}\left(\frac{g'}{2g} - \frac{h'}{2h} - \frac{x}{2} + \frac{\mathcal{E}_1 - \mathcal{E}_2}{g}\right) - \mathcal{E}_2 + \frac{\mathcal{E}\ \ + \mathcal{E}_4}{2}\right] \\ & \times \exp\left[\int \left(\frac{g'}{2g} + \frac{g}{2} + \frac{\mathcal{E}_1 - \mathcal{E}_2}{g}\right) \mathrm{d}x\right], \end{aligned} \tag{27}$$

$$\psi_{\mathcal{E}\ \ } \propto \exp\left[\int \left(\frac{h'}{2h} + \frac{h}{2} + \frac{\mathcal{E}_4 - \mathcal{E}\ \ }{h}\right) \mathrm{d}x\right], \tag{28}$$

$$\psi_{\mathcal{E}_4} \propto \exp\left[\int \left(\frac{h'}{2h} + \frac{h}{2} + \frac{\mathcal{E}\ \ - \mathcal{E}_4}{h}\right) \mathrm{d}x\right]. \tag{29}$$

Thus, in the direct approach given the PV solution $w(z)$ all the relevant functions of the system turn out to be determined.

On the other hand, in the inverse approach one looks for systems ruled by third degree PHA, in particular their four extremal states and associated factorization energies. Now, $h(x)$ is found from equations (28)–(29), which in turn is directly related with the solution $w(z)$ to the PV equation as follows:

$$h(x) = \frac{2(\mathcal{E}\ \ - \mathcal{E}_4)}{[\ln(\psi_{\mathcal{E}_4}) - \ln(\psi_{\mathcal{E}\ \ })]'} = \{\ln[W(\psi_{\mathcal{E}\ \ }, \psi_{\mathcal{E}_4})]\}', \tag{30}$$

$$w(z) = \frac{h(\sqrt{z})}{\sqrt{z} + h(\sqrt{z})}. \tag{31}$$

Thus, the extremal states of systems ruled by PHA of third degree produce solutions to the PV equation through equations (30)–(31).

5. Radial oscillator SUSY partners

The Hamiltonian for the radial oscillator reads [7]

$$H_\ell = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_\ell(x) = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{x^2}{8} + \frac{\ell(\ell+1)}{2x^2}, \tag{32}$$

which has second order differential ladder operators given by

$$b_\ell^\pm = \frac{1}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} \mp x\frac{\mathrm{d}}{\mathrm{d}x} + \frac{x^2}{\ } - \frac{\ell(\ell+1)}{x^2} \mp \frac{1}{2}\right), \tag{33}$$

so they generate a first degree PHA:

$$[H_\ell, b_\ell^\pm] = \pm b_\ell^\pm, \tag{34}$$

$$[b_\ell^-, b_\ell^+] = 2H_\ell. \tag{35}$$

The analogue of the number operator reads

$$b_\ell^+ b_\ell^- = (H_\ell - \mathcal{E}_1)(H_\ell - \mathcal{E}_2) = \left(H_\ell - \frac{\ell}{2} - \frac{3}{\ }\right)\left(H_\ell + \frac{\ell}{2} - \frac{1}{\ }\right), \tag{36}$$

while the two extremal states, which are annihilated by b_ℓ^- and are formal eigenfunctions of H_ℓ, are given by

$$\psi_{\mathcal{E}_1} \propto x^{\ell+1}\exp(-x^2/\ \ ), \quad \mathcal{E}_1 = \frac{\ell}{2} + \frac{3}{\ } \equiv E_{0\ell}, \tag{37}$$

$$\psi_{\mathcal{E}_2} \propto x^{-\ell}\exp(-x^2/\ \ ), \quad \mathcal{E}_2 = -\frac{\ell}{2} + \frac{1}{\ } = -E_{0\ell} + 1. \tag{38}$$

Only the first one fulfills the boundary conditions and thus leads to a ladder of physical eigenfunctions. The spectrum of the radial oscillator is therefore

$$\mathrm{Sp}(H_\ell) = \left\{E_{n\ell} = n + \frac{\ell}{2} + \frac{3}{\ },\ n = 0, 1, \ldots\right\}. \tag{39}$$

Now, a kth order SUSY transformation is applied to $V_\ell(x)$ by using k seed solutions of the form

$$\begin{aligned} u(x,\epsilon) = {} & x^{-\ell}\mathrm{e}^{-\frac{x^2}{4}}\left[{}_1F_1\left(\frac{1-2\ell-\ \ \epsilon}{\ }, \frac{1-2\ell}{2}; \frac{x^2}{2}\right)\right. \\ & \left. + \ \nu\frac{\Gamma\left(\frac{\ +2\ell-4\epsilon}{4}\right)}{\Gamma\left(\frac{\ +2\ell}{2}\right)}\left(\frac{x^2}{2}\right)^{\ell+1/2} {}_1F_1\left(\frac{3+2\ell-\ \ \epsilon}{\ }, \frac{3+2\ell}{2}; \frac{x^2}{2}\right)\right], \end{aligned} \tag{40}$$

which creates k new levels below $E_{0\ell}$ as follows:

$$\epsilon_k < \epsilon_{k-1} < \cdots < \epsilon_1 < E_{0\ell}, \tag{41}$$

where $\nu_j \geq -\frac{\Gamma\left(\frac{1-2\ell}{2}\right)}{\Gamma\left(\frac{1-2\ell-4\epsilon}{4}\right)}$ for j odd and $\nu_j \leq -\frac{\Gamma\left(\frac{1-2\ell}{2}\right)}{\Gamma\left(\frac{1-2\ell-4\epsilon}{4}\right)}$ for j even.

The new potential reads now

$$V_k(x) = \frac{x^2}{8} + \frac{\ell(\ell+1)}{2x^2} - \{\ln[W(u_1,\ldots,u_k)]\}'', \tag{42}$$

and the associated spectrum is

$$\mathrm{Sp}(H_k) = \{\epsilon_k, \ldots, \epsilon_1, E_{0\ell}, E_{1\ell}, \ldots\}. \tag{43}$$

The natural $(2k+2)$th-order ladder operators of H_k are given by

$$L_k^\pm = B_k^+ b_\ell^\pm B_k^-, \tag{44}$$

which satisfy

$$[H_k, L_k^\pm] = \pm L_k^\pm, \tag{45}$$

while the analogue of the number operator reads

$$N(H_k) = \left(H_k - \frac{\ell}{2} - \frac{3}{\ }\right)\left(H_k + \frac{\ell}{2} - \frac{1}{\ }\right)\prod_{j=1}^{k}(H_k - \epsilon_j)(H_k - \epsilon_j - 1). \tag{46}$$

In order to link with the PV equation, a reduction for the order of the ladder operators from $2k+2$ to   is required, which is achieved when the requirements contained in the following theorem are fulfilled.

Theorem 1. *Let $V_k(x)$, the SUSY partners of $V_\ell(x)$, be generated by k seed solutions which are connected as follows:*

$$u_j = (b_\ell^-)^{j-1}u_1, \quad \epsilon_j = \epsilon_1 - (j-1), \quad j = 1, \ldots, k, \tag{47}$$

where the free solution u_1 is nodeless, $\epsilon_1 < E_0 = \frac{\ell}{2} + \frac{\ }{4}$, $\nu_1 \geq -\frac{\Gamma\left(\frac{1-2\ell}{2}\right)}{\Gamma\left(\frac{1-2\ell-4\epsilon_1}{4}\right)}$. Then,

$$L_k^+ = P_{k-1}(H_k)\mathbf{l}_k^+, \tag{48}$$

$$P_{k-1}(H_k) = (H_k - \epsilon_1)\cdots(H_k - \epsilon_{k-1}), \tag{49}$$

where $\mathbf{l}_k^+$ is a fourth-order differential ladder operator such that

$$[H_k, \mathbf{l}_k^+] = \mathbf{l}_k^+, \tag{50}$$

$$\mathbf{l}_k^+\mathbf{l}_k^- = (H_k - E_0)(H_k + E_0 - 1)(H_k - \epsilon_k)(H_k - \epsilon_1 - 1). \tag{51}$$

The proof of this theorem can be found in [7]. Let us note that the operators $\{H_k, \mathbf{l}_k^-, \mathbf{l}_k^+\}$ satisfy a third degree PHA, which imply that solutions to the PV equation can be found from the corresponding extremal states

$$\psi_{\mathcal{E}_1} \propto B_k^+ b_\ell^+ u_1, \qquad \mathcal{E}_1 = \epsilon_1 + 1, \tag{52}$$

$$\psi_{\mathcal{E}_2} \propto B_k^+\left[x^{-\ell}\exp(-x^2/\ )\right], \qquad \mathcal{E}_2 = -E_0 + 1, \tag{53}$$

$$\psi_{\mathcal{E}} \propto \frac{W(u_1, \ldots, u_{k-1})}{W(u_1, \ldots, u_k)}, \qquad \mathcal{E} = \epsilon_k, \tag{54}$$

$$\psi_{\mathcal{E}_4} \propto B_k^+\left[x^{\ell+1}\exp(-x^2/\ )\right], \qquad \mathcal{E}_4 = E_0. \tag{55}$$

If we remember that the PV transcendent is generated from equations (30)–(31), making all possible permutations on the ordering of the extremal states, we found the solutions for $k = 1$, $\epsilon_1 = E_{0\ell}$ and $k = 2$, $\epsilon_1 = E_{1\ell}$ shown in Tables 1 and 2, respectively.

Order	$w(z)-1$
123	0
132	0
1 23	$z(2\ell - z + 1)^{-1}$
231	0
2 13	$\frac{z[8\ell\ \ -4\ell^2(z-1)-2\ell(5z^2+2z+2)+5(z-\ \ )z^2]}{-8\ell\ \ (z-4)+4\ell^2(z^2-\ \ z+4)+2\ell(5z\ \ +2z^2-2z-8)-5(z-1)z}$
3 12	$\frac{z[8\ell\ \ +4\ell^2-2\ell(2+5z^2)-15z^2]}{16\ell(2\ell^2+\ell-1)}$

Table 1. PV transcendents for $k = 1, \epsilon_1 = E_{0\ell}$.

Order	$w(z)$
123	0
132	0
1 23	$\frac{4(z-2\ell-\ \ )}{z^2-2z(2\ell+1)+4\ell^2+8\ell+}$
231	0
2 13	$\frac{(-z+2\ell+\ \ )(2\ell+1)}{z^2-2z(2\ell+1)+4\ell(\ell-2)+}$
3 12	∞

Table 2. PV transcendents for $k = 2, \epsilon_1 = E_{1\ell}$.

We can obtain also analytic expressions when using the general solution as given in (40). However, if we choose seed solutions with factorization energies that do not coincide with the eigenvalues of H_ℓ, the explicit expressions for the PV transcendents become too long to be written here. Hence, we decided just to plot some of these solutions. Thus, in Fig. 1(left) the PV transcendents generated from first-order SUSY ($k = 1$) for $\ell = 1$, $\epsilon_1 = 1$ and different values of the parameter ν_1 are shown. In Fig. 1(right) the corresponding solutions appearing from second-order SUSY ($k = 2$) for $\ell = 0$, $\nu_1 = 0$ and different values of the factorization energy ϵ_1 are plotted. A complex PV transcendent, generated from a complex first-order SUSY ($k = 1$) for $\ell = 2$, $\epsilon_1 = 2$, $\nu_1 = i$ is shown in Fig. 2 (see, e.g., [10, 13]).

It is important to note that the PV transcendents derived through the general solution to the stationary Schrödinger equation belong to the so-called confluent hypergeometric function hierarchy. It is possible to make a more detailed classification, by taking specific values of ϵ_1 and/or ν_1. In this way the following hierarchies of solutions can be obtained:

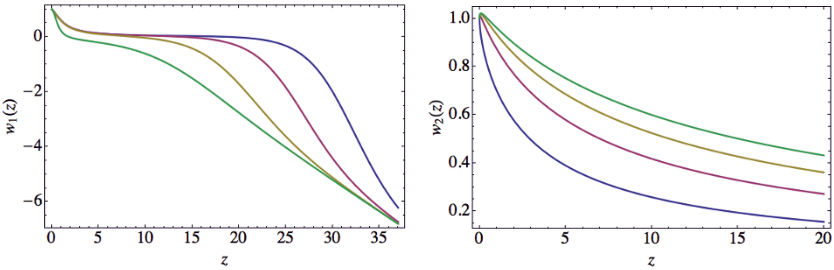

FIGURE 1. Real PV transcendents generated through first-order SUSY for $\ell = 1$, $\epsilon_1 = 1$, $\nu_1 = \{0.905$ (blue), 0.913 (magenta), 1 (yellow), 10 (green)$\}$ (left), and through second-order SUSY for $\ell = 0$, $\nu_1 = 0$, $\epsilon_1 = \{1/\ $ (blue), $-3/\ $ (magenta), $-7/\ $ (yellow), $-11/\ $ (green)$\}$ (right).

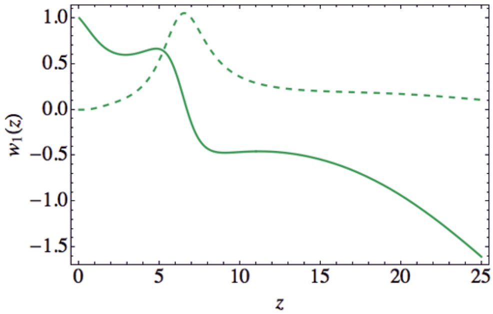

FIGURE 2. Real (continuous curve) and imaginary part (dashed curve) of the complex PV transcendent generated through first-order SUSY for $\ell = 2$, $\epsilon_1 = 2$ and $\nu_1 = i$.

Laguerre polynomial hierarchy. For some particular values of ϵ_1 the confluent hypergeometric functions in the general seed solution become the Laguerre polynomials. Two examples of such solutions are given by:

$$w_1(z) = \ 1 - z^{-1/2} L_0^{(\alpha)}, \tag{56}$$

$$w_1(z) = \ 1 - \frac{z^{\ /2} L_1^{(\alpha)}(z^2/2)}{2L_1^{(\alpha)}(z^2/2) - 2\alpha - 1}, \quad \alpha = -(2\ell + 1)/2. \tag{57}$$

Hermite polynomial hierarchy. For certain values of ϵ_1 the confluent hypergeometric functions becomes a Hermite polynomial; two members of such a hierarchy are:

$$w_1(z) = 1 - \frac{z^{\ /2} H_{2n}(z)}{(z^2+1)H_{2n}(z) - \ \ nzH_{2n-1}(z)}, \tag{58}$$

$$w_1(z) = 1 + \frac{z^{1/2} H_{2n}(z)}{nH_{2n-1}(z) - zH_{2n}(z)}. \tag{59}$$

6. Conclusions

In this paper we have stressed the importance that systems ruled by polynomial Heisenberg algebras has in theoretical and mathematical physics. In particular, it was pointed out that the third degree PHA is connected with PV equation. Moreover, it was shown that SUSY QM applied to the radial oscillator supplies the simplest realizations of the third degree PHA.

On the other hand, by using inverse techniques it was possible to design an algorithm for generating solutions to the PV equation. The simplest PV transcendents were also generated through such recipe (see [8]).

We believe a greater effort is required to improve the present classification of the PV hierarchies [7]. We hope to contribute to this task in the near future, as we did previously in the case of Painlevé IV transcendents (see [14, 15]).

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Hamiltonian Dynamics for the Kepler Problem in a Deformed Phase Space

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Abstract. This work addresses the Hamiltonian dynamics of the Kepler problem in a deformed phase space, by considering the equatorial orbit. The recursion operators are constructed and used to compute the integrals of motion. The same investigation is performed with the introduction of the Laplace–Runge–Lenz vector. The existence of quasi-bi-Hamiltonian structures is also elucidated. Related properties are studied.

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Keywords. Hamiltonian dynamics, Kepler problem, deformed phase space, Laplace–Runge–Lenz vector, quasi-bi-Hamiltonian structure.

1. Introduction

In 1601, Kepler obtained a detailed set of observations of the motion of the planet Mars from the Danish astronomer Tycho Brahe [4]. From his analysis of these data, Kepler determined that the path of Mars is an ellipse, with the Sun located at a focal point, and that the radius vector from the Sun to the planet sweeps out equal areas in equal times. The direct problem was to determinate the nature of the force required to maintain elliptical motion about a focal force center. This direct problem remained unsolved until after 1679, when Newton determined the functional dependence on distance of the force required to sustain such an elliptical path of Mars about the Sun as a center of force located at a focal point of the ellipse.

Building on Newton's description of the nature and universality of the gravitational force, scientists of the eighteenth century shifted their interest almost exclusively from direct to inverse problems. They used the combined gravitational forces of the Sun and the other planets to predict and explain perturbations in the conic paths of planets and comets. That interest continued through the nineteenth and twentieth centuries, and today scientists still concentrate upon the inverse problem rather than the direct one.

In particular, in the last few decades there was a renewed interest in the Kepler problem as one of completely integrable Hamiltonian systems (IHS), the concept of which goes back to Liouville in 1897 [19] and Poincaré in 1899 [24]. Loosely speaking, IHS are dynamical systems admitting a Hamiltonian description, and possessing sufficiently many constants of motion. Many of these systems are Hamiltonian systems with respect to two compatible symplectic structures [11, 12, 20, 33] leading to a geometrical interpretation of the so-called recursion operator [18]. The theory of integrable Hamiltonian systems, based on the use of the Nijenhuis torsion, is a part of the geometry of a particular class of manifolds, called Poisson–Nijenhuis manifolds [21]. In 1992, Marmo and Vilasi [23] constructed a recursion operator for the Kepler dynamics, and obtained related constants of motion.

From the Magri works [20, 21], it is known that the eigenvalues of the recursion operator of bi-Hamiltonian systems form a set of pairwise Poisson-commuting invariants [6]. It is, however, worth noticing that two kinds of difficulties often arise, while investigating these systems: (*i*) Firstly, it is in general very difficult to give locally an explicit second Hamiltonian structure for a given integrable Hamiltonian system [25] even if it is theoretically always possible in the neighborhood of a regular point of the Hamiltonian [7]; (*ii*) Secondly, the global or semi-local existence of such structures implies very strong conditions which are rarely satisfied [8, 10].

In 1996, R. Brouzet et al. defined a weaker notion under the name of *quasi-bi-Hamiltonian system* (QBHS) which relaxes these two difficulties for two degrees of freedom. In 2000, G. Sparano et al. constructed recursion operator for the Kepler dynamics, in the non-commutative case using the so-called Delauney action-angle coordinates [28]. Further, in 2013, Hosokawa and Takeuchi [15] solved the same problem, but using the Runge–Lenz–Pauli vector, and got new constants of motion. A bi-Hamiltonian formulation for a Kepler problem was also studied with Delaunay-type variables [14]. In 2016, J. F. Cariñena et al. [9] investigated some properties of the Kepler problem related to the existence of quasi-bi-Hamiltonian structures. In this work, we investigate the Kepler dynamics in a deformed phase space.

The paper is organized as follows. In Section 2, we present the considered deformed phase space. In Section 3, we define, in action-angle coordinates, the deformed Hamiltonian function, symplectic form and vector field describing the Kepler dynamics. In Section 4, we construct recursion operators, and compute the associated integrals of motion. In Section 5, we give an alternative Hamiltonian description for the dynamical systems and obtain associated recursion operators in a non resonant case. In Section 6, we study the existence of quasi-bi-Hamiltonian structure for the considered Kepler dynamics. In Section 7, we end with some concluding remarks.

2. Deformed phase space and Kepler Hamiltonian

Let $\mathbb{R}^3_0 = \mathbb{R}^3 \backslash \{0,0,0\}$ be the configuration manifold $\mathcal{Q}$, and $\mathcal{T}^*\mathcal{Q} = \mathcal{Q} \times \mathbb{R}^3$ be the cotangent bundle with the local coordinates (q,p). The cotangent bundle $\mathcal{T}^*\mathcal{Q}$ has a natural symplectic structure ω which, in local coordinates, is given by

$$\omega = \sum_{i=1}^{3} dq^i \wedge dp_i.$$

Since ω is non-degenerate, it induces the map Λ: $\mathcal{T}^*\mathcal{Q} \longrightarrow \mathcal{T}\mathcal{Q}$ defined by

$$\Lambda = \sum_{i=1}^{3} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i},$$

where $\mathcal{T}\mathcal{Q}$ is the tangent bundle. The map Λ is called the bivector field [34] and used to construct the Hamiltonian vector field X_f of a Hamiltonian function f by the relation

$$X_f = \Lambda df. \tag{1}$$

The phase space deformation is here understood by replacing the usual product with the γ-star product (also known as the Moyal product law) between two arbitrary functions of position and momentum [16, 22, 32]:

$$(f *_\gamma g)(q,p) = f(q_i,p_i) \exp\left(\frac{1}{2}\gamma^{ab}\overleftarrow{\partial}_a \overrightarrow{\partial_b}\right) g(q_j,p_j)\Big|_{(q_i,p_i)=(q_j,p_j)}, \tag{2}$$

where

$$\gamma^{ab} = \begin{pmatrix} \Theta^{ij} & \delta^i_j \\ -\delta^i_j & 0 \end{pmatrix}, \tag{3}$$

Θ is an antisymmetric $n \times n$ matrix inducing the deformation in the coordinates. Without loss of generality, we restrict our study to the first two terms of the $*_\gamma$ deformed Poisson bracket expansion to obtain

$$\{f,g\}_\gamma = \Theta^{ij}\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial q^j} + \left(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}\right), \tag{4}$$

giving

$$\{q^i,q^j\}_\gamma = \Theta^{ij}, \quad \{q^i,p_j\}_\gamma = \delta^i_j, \quad \{p_i,p_j\}_\gamma = 0. \tag{5}$$

The Kepler Hamiltonian in $\mathcal{T}^*\mathcal{Q}$ takes the form

$$H = \frac{p_i p^i}{2m} + V(r), \tag{6}$$

yielding the Hamilton's equations:

$$\dot{q}^i := \{q^i,H\}_\gamma = \frac{p^i}{m} + \Theta^{ij}\frac{\partial V(r)}{\partial q^j}, \quad \dot{p}_i := \{p_i,H\}_\gamma = -\frac{\partial V(r)}{\partial q^i}, \tag{7}$$

and the following correction to the Newton second law [27]:

$$m\ddot{q}^i = -\frac{q^i}{r}\frac{k}{r^2} + m\varepsilon^{ijk}\dot{q}^j\Omega^k + m\varepsilon^{ijk}q^j\dot{\Omega}^k, \tag{8}$$

where the deformation parameter $\Theta^{ij} = \varepsilon^{ijk}\alpha^k$, and the angular velocity

$$\Omega^i = \frac{k}{r^3}\alpha^i, \quad i = 1, 2, 3.$$

Setting the deformation parameter $\alpha^i = \delta^{i3}\alpha$ transforms H into

$$H = \frac{1}{2}\left[(\dot{q}^1 - q^2\Omega)^2 + (\dot{q}^2 + q^1\Omega)^2 + (\dot{q}^3)^2\right] - \frac{k}{r}, \tag{9}$$

which is reduced to

$$H = \frac{p_r^2}{2m} + \frac{p_{\varphi_\alpha}^2}{2mr^2} - \frac{k}{r} \tag{10}$$

in spherical coordinates (r, υ, φ), and equatorial orbit corresponding to $\upsilon = \frac{\pi}{2}$, where $p_r = m\dot{r}$ and $p_{\varphi_\alpha} = mr^2\dot{\varphi}_\alpha$, with $\dot{\varphi}_\alpha = (\dot{\varphi}+\Omega)$ and $\varphi_\alpha = (\varphi+\Omega t) \in (0, 2\pi)$.

Equation (9) encodes the information on the phase space deformation through Ω, which depends on the deformation parameter α. However, it can evidently be interpreted as equivalent to the Hamiltonian for a charged particle in a homogeneous, independent of time, magnetic field along z axis, and the central Newtonian gravitational field in the usual commutative space.

Now considering the coordinate system $(r, \varphi_\alpha, p_r, p_{\varphi_\alpha})$, and using (1), we get the following Hamiltonian vector field:

$$X_H = \frac{1}{m}\left[p_r\frac{\partial}{\partial r} - \frac{1}{mr^3}\left(- p_{\varphi_\alpha}^2 + mkr\right)\frac{\partial}{\partial p_r} + \frac{p_{\varphi_\alpha}}{mr^2}\frac{\partial}{\partial \varphi_\alpha}\right]. \tag{11}$$

3. Hamiltonian system in the action-angle coordinates

The Hamiltonian function (10) does not explicitly depend on the time. Then, setting $V = W - Et$, it is possible to find a complete integral for the equation of motion by using the method of variable separation:

$$W = W_r(r) + W_{\varphi_\alpha}(\varphi_\alpha). \tag{12}$$

In this case, the Hamilton–Jacobi equation [3] is reduced to

$$E = \frac{1}{2m}\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial W}{\partial \varphi_\alpha}\right)^2 - \frac{k}{r}, \tag{13}$$

leading to the following set of equations:

$$\begin{cases} \left(\dfrac{dW_{\varphi_\alpha}(\varphi_\alpha)}{d\varphi_\alpha}\right)^2 = D_{\varphi_\alpha}^2, \\ -r^2\left(\dfrac{dW_r(r)}{dr}\right)^2 + 2mr^2E + 2mrk = D_{\varphi_\alpha}^2, \end{cases}$$

where D_{φ_α} is constant. In the compact case [34], characterized by $E < 0$, we can introduce the action variables [2] J_r and J_{φ_α} such that

$$\begin{cases} J_{\varphi_\alpha} = \dfrac{1}{2\pi} \oint \dfrac{dW_{\varphi_\alpha}(\varphi_\alpha)}{d\varphi_\alpha} d\varphi_\alpha, \\ J_r = \dfrac{1}{2\pi} \oint \dfrac{dW_r(r)}{dr} dr. \end{cases}$$

Using the method of residues [1, 34], we get

$$J_r = -p_{\varphi_\alpha} + \frac{mk}{\sqrt{-2mE}}, \quad D_{\varphi_\alpha} = p_{\varphi_\alpha},$$

and the integrable system [5]:

$$\begin{cases} \dot{J}_i = 0, \\ \dot{\varphi}^i = \dfrac{\partial H}{\partial J_i} \end{cases} \implies \begin{cases} llJ_1 = J_r; \;\; J_2 = J_{\varphi_\alpha}, \\ \varphi^1 = \dfrac{mk^2}{(J_1+J_2)^3} t; \;\; \varphi^2 = \dfrac{mk^2}{(J_1+J_2)^3} t, \;\; \varphi^1(0) = \varphi^2(0) = 0. \end{cases} \tag{14}$$

Proposition 1. *In action-angle coordinates (J, φ), Hamiltonian H, symplectic form ω, and the Hamiltonian vector field X_H are respectively:*

$$H = E = -\frac{mk^2}{2(J_1+J_2)^2}, \quad \omega = \sum_{h=1}^{2} dJ_h \wedge d\varphi^h, \tag{15}$$

and

$$X_H = \{H, \cdot\} := \frac{mk^2}{(J_1+J_2)^3} \left(\frac{\partial}{\partial \varphi^1} + \frac{\partial}{\partial \varphi^2} \right), \tag{16}$$

where $\{\cdot, \cdot\}$ is the usual Poisson bracket.

4. Recursion operators

Let us define a 2-form ω_1 and a vector field Δ,

$$\omega_1 := \sum_{h,k=1}^{2} S_h^k dJ_k \wedge d\varphi^h = \sum_{h=1}^{2} d\lambda_h \wedge d\varphi^h, \quad \Delta := \lambda_h \frac{\partial}{\partial J_h}, \tag{17}$$

where $S = \begin{pmatrix} J_1 & J_2 \\ J_2 & J_1 \end{pmatrix}$, and $\begin{cases} \lambda_1 = \dfrac{1}{2}\big(J_1^2 + J_2^2\big), \\ \lambda_2 = J_2 J_1, \end{cases}$ are such that ω_1 is the Lie derivative of the symplectic form ω in (15) with respect to the vector field Δ, i.e.,

$$\mathcal{L}_\Delta \omega = \omega_1.$$

The vector field Δ generates a sequence of finitely many (Abelian) symmetries according to the following scheme:

$$X_{i+1} := [X_i, \Delta]_\mu = \frac{2}{\mu}(X_i(\Delta) - \Delta(X_i)),$$

where $\mu = 3 - i,\ i = 0, 1, 2$ and $X_0 = X_H$ in (16). The X_i's, given by

$$X_0 = \frac{mk^2}{(J_1+J_2)^3}\left(\frac{\partial}{\partial\varphi^1} + \frac{\partial}{\partial\varphi^2}\right), \quad X_1 = \frac{mk^2}{(J_1+J_2)^2}\left(\frac{\partial}{\partial\varphi^1} + \frac{\partial}{\partial\varphi^2}\right), \tag{18}$$

$$X_2 = \frac{mk^2}{(J_1+J_2)}\left(\frac{\partial}{\partial\varphi^1} + \frac{\partial}{\partial\varphi^2}\right), \quad X_3 = mk^2\left(\frac{\partial}{\partial\varphi^1} + \frac{\partial}{\partial\varphi^2}\right), \tag{19}$$

are:

(i) in involution, i.e.,

$$[X_h, X_k]_\mu = 0, \quad h, k = 0, 1, 2, 3, \quad \mu = 1, 2, 3. \tag{20}$$

(ii) Hamiltonian vector fields, i.e., can be expressed as:

$$X_i = \{H_i, \cdot\} = \{H_{i+1}, \cdot\}_1, \quad i = 0, 1, 2, \tag{21}$$

with respect to the Poisson bracket $\{\cdot,\cdot\}_1$ defined by

$$\{f, g\}_1 := \sum_{h,k=1}^{n} (S^{-1})^h_k \left(\frac{\partial f}{\partial J_k}\frac{\partial g}{\partial \varphi^h} - \frac{\partial f}{\partial \varphi^h}\frac{\partial g}{\partial J_k}\right), \tag{22}$$

where

$$S^{-1} = \begin{pmatrix} \dfrac{J_1}{(J_1-J_2)(J_1+J_2)} & \dfrac{-J_2}{(J_1-J_2)(J_1+J_2)} \\ \dfrac{-J_2}{(J_1-J_2)(J_1+J_2)} & \dfrac{J_1}{(J_1-J_2)(J_1+J_2)} \end{pmatrix},$$

and

$$H_0 = \frac{-mk^2}{2(J_1+J_2)^2}, \qquad H_1 = \frac{-mk^2}{(J_1+J_2)},$$
$$H_2 = mk^2 \ln(J_1+J_2), \qquad H_3 = mk^2(J_1+J_2). \tag{23}$$

Proposition 2. *The recursion operator for the Kepler dynamics in the action-angle coordinates (J, φ) is given by*

$$T = \sum_{h,k} (S)^h_k \left(\frac{\partial}{\partial J_h} \otimes dJ_k + \frac{\partial}{\partial \varphi_h} \otimes d\varphi_k\right), \ \textit{where} \ \ S = \begin{pmatrix} J_1 & J_2 \\ J_2 & J_1 \end{pmatrix},$$

$\mathcal{L}_{X_l} T = 0$ $(l = 0, 1, 2, 3)$, *and the Nijenhuis torsion vanishes, i.e.,*

$$(\mathcal{N}_T)^h_{ij} := T^k_i \frac{\partial T^h_j}{\partial J^k} - T^k_j \frac{\partial T^h_i}{\partial J^k} + T^h_k \frac{\partial T^k_i}{\partial J^j} - T^h_k \frac{\partial T^k_j}{\partial J^i} = 0 \quad (i, j, k, h = 1, 2).$$

Consider the constants of motion [26],

$$H \text{ in (15)}, \quad M = mr^2(\dot{\varphi} + 2\Omega), \quad \text{and} \quad L_\alpha = M + m\alpha H,$$

i.e.,

$$\{H, M\} = 0, \quad \{H, L_\alpha\} = 0, \quad \{M, L_\alpha\} = 0. \tag{24}$$

Then, there exist functions ϕ_1, ϕ_2 satisfying

$$\omega' = d\xi_1 \wedge d\phi_1 + d\xi_2 \wedge d\phi_2, \tag{25}$$

such that the equations of motion in the coordinate system (ξ, ϕ) are

$$\begin{cases} \dot{\xi}_i = 0, \\ \dot{\phi}^i = \dfrac{\partial H'}{\partial \xi_i} \end{cases} \Longrightarrow \begin{cases} \xi_i = cst, \\ \phi^i = \dfrac{\partial H'}{\partial \xi_i} t, \quad \phi^i(0) = 0, \;\; i \in \{1, 2\}, \end{cases} \tag{26}$$

where $\xi_1 = L_\alpha$, $\xi_2 = M$, $H' = \dfrac{1}{m\alpha}(\xi_1 - \xi_2)$. We get the relationships

$$J_1 = -\xi_2 + \varpi + \sqrt{\frac{m^2 \alpha k^2}{2(\xi_2 - \xi_1)}}, \quad J_2 = \xi_2 - \varpi, \tag{27}$$

$$\varphi^1 = \frac{2\sqrt{2\alpha}}{m\alpha k}(\xi_2 - \xi_1)^{3/2}\phi^1, \qquad \varphi^2 = -\frac{2\sqrt{2\alpha}}{m\alpha k}(\xi_2 - \xi_1)^{3/2}\phi^2, \tag{28}$$

where $\varpi = mr^2 \Omega$, $\xi_2 > \xi_1 > 0, \alpha > 0$. Finally, we arrive at

Proposition 3. *In the coordinate system (ξ, ϕ), Hamiltonian function H', symplectic form ω', Hamiltonian vector field X'_H, and the recursion operator T' are respectively:*

$$H' = \frac{1}{m\alpha}(\xi_1 - \xi_2), \quad \omega' = \sum_{h=1}^{2} d\xi_h \wedge d\phi^h, \quad X'_{H'} = \frac{1}{m\alpha}\left(\frac{\partial}{\partial \phi_1} - \frac{\partial}{\partial \phi_2}\right), \tag{29}$$

$$T' = \sum_{i=1}^{2} R_i \left(\frac{\partial}{\partial \xi_i} \otimes d\xi_i + \frac{\partial}{\partial \phi_i} \otimes d\phi_i\right), \;\; \textit{where } R = \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix}. \tag{30}$$

Two interesting cases deserve investigation:

I) Introduce the Laplace–Runge–Lenz (LRL) vector A given by [13]

$$A = p \times L - mk\frac{q}{r}, \tag{31}$$

where p is the momentum vector, q is the position vector of the particle of mass m, and L is the angle momentum vector, $L = q \times p$ [29]. We obtain

$$L_1 = 0, \;\; L_2 = 0, \;\; L_3 = mr^2 \dot{\varphi}_\alpha = p_{\varphi_\alpha}, \tag{32}$$

$$A_1 = C\sin\beta + D\cos\beta; \;\; A_2 = C\cos\beta - D\sin\beta; \;\; A_3 = 0, \tag{33}$$

$$\begin{aligned} \{A_1, H\} &:= \left(\frac{\partial A_1}{\partial r}\frac{\partial H}{\partial p_r} - \frac{\partial A_1}{\partial p_r}\frac{\partial H}{\partial r}\right) + \left(\frac{\partial A_1}{\partial \varphi_\alpha}\frac{\partial H}{\partial p_{\varphi_\alpha}} - \frac{\partial A_1}{\partial p_{\varphi_\alpha}}\frac{\partial H}{\partial \varphi_\alpha}\right) \\ &= \frac{3k\alpha p_r}{mr^4}(D\sin\beta - B\cos\beta), \end{aligned} \tag{34}$$

$$\{A_2, H\} = \frac{3k\alpha p_r}{mr^4}(B\sin\beta - D\cos\beta), \tag{35}$$

where

$$C = -p_r p_{\varphi_\alpha}\cos\varphi_\alpha + \frac{p_{\varphi_\alpha}^2}{r}\sin\varphi_\alpha - mk\sin\varphi_\alpha$$

and

$$D = p_r p_{\varphi_\alpha} \sin\varphi_\alpha + \frac{p_{\varphi_\alpha}^2}{r}\cos\varphi_\alpha - mk\cos\varphi_\alpha, \quad \beta = \Omega t.$$

Remark 4. We have:

(i) The A_i, $i = 1,2,3$, commute with the Hamiltonian H in (10), i.e., $\{A_i, H\} = 0$, if

$$\beta = \frac{\pi}{4}; \quad \frac{p_r p_{\varphi_\alpha}}{\frac{p_{\varphi_\alpha}^2}{r} - mk} = -\cot(\beta + \varphi_\alpha), \ \ (\beta+\varphi_\alpha) \in (0,\pi). \tag{36}$$

(ii) $\{A_1, A_2\} = \Big(-2mH + \dfrac{3k\alpha p_r}{r^4}\Big)p_{\varphi_\alpha}$, $\{A_1, L_3\} = A_2$, and $\{A_2, L_3\} = A_1$.

(iii) Setting $L_3 = A_3$ and $p_{\varphi_\alpha}^2 = r[2mk - r + p_r(3\Omega - rp_r)] \equiv A_3^2$, then, the $A_i{}'s$ generate an $su(2)$ Lie algebra, i.e., $\{A_i, A_j\} = \varepsilon_{ijl}A_l$.

II) Consider a scaled Runge–Lenz–Pauli vector Γ, defined on the domain $\{(q,p) \in \mathcal{T}^*(\mathbb{R}^3\backslash\{0,0,0\}) | H(q,p) < 0\}$ by

$$\Gamma = \frac{1}{\sqrt{-2mH}}A, \tag{37}$$

where H is the Hamiltonian function given in (10). The components Γ_i are

$$\begin{aligned}\Gamma_1 &= \frac{1}{\sqrt{-2mH}}(C\sin\beta + D\cos\beta),\\ \Gamma_2 &= \frac{1}{\sqrt{-2mH}}(C\cos\beta - D\sin\beta), \quad \Gamma_3 = 0,\end{aligned} \tag{38}$$

with

$$|\Gamma|^2 = -\frac{mk^2}{2H} + L_3^2. \tag{39}$$

The quantities H, $|\Gamma|^2$, and L_3 are in involution, i.e.,

$$\{|\Gamma|^2, L_3\} = 0, \quad \{|\Gamma|^2, H\} = 0, \quad \{L_3, H\} = 0.$$

Putting $\pi_1 = |\Gamma|^2$ and $\pi_2 = p_{\varphi_\alpha}$, the equations of motion in the (π,χ) system become:

$$\begin{cases}\dot{\pi}_i = 0,\\ \dot{\chi}^i = \dfrac{\partial H''}{\partial \pi_i}, \quad H'' = \dfrac{mk^2}{2(\pi_2^2 - \pi_1)}\end{cases} \Longrightarrow \begin{cases}\pi_i = cst, \quad i = 1,2,\\ \chi^i = \dfrac{\partial H''}{\partial \xi_i}t + \chi^i(0), \quad \chi^i(0) = 0.\end{cases} \tag{40}$$

The relationships between (J,φ) and (π,χ) are deduced as:

$$J_1 = -\pi_1 + \sqrt{\pi_1 - \pi_2^2}, \quad J_2 = \pi_2, \quad \chi^1 = \frac{1}{(J_1+J_2)}\varphi^1, \quad \chi^2 = -\frac{J_2}{(J_1+J_2)}\varphi^2. \tag{41}$$

Finally, we get

Proposition 5. *In the coordinate system* (π, χ), *Hamiltonian function* H'', *symplectic form* ω'', *Hamiltonian vector field* $X''_{H''}$, *and the recursion operator* T'' *are given as follows:*

$$H'' = \frac{mk^2}{2(\pi_2^2 - \pi_1)}, \quad \omega'' = \sum_{h=1}^{2} d\pi_h \wedge d\chi^h, \tag{42}$$

$$X''_{H''} = \frac{mk^2}{2(\pi_2^2 - \pi_1)^2}\left(\frac{\partial}{\partial \chi^1} - 2\pi_2 \frac{\partial}{\partial \chi^2}\right), \tag{43}$$

$$T'' = \sum_{i=1}^{2} F_i \left(\frac{\partial}{\partial \pi_i} \otimes d\pi_i + \frac{\partial}{\partial \chi^i} \otimes d\chi^i\right), \ \textit{where} \ \ F = \begin{pmatrix} \pi_1 & 0 \\ 0 & \pi_2 \end{pmatrix}. \tag{44}$$

5. Alternative Hamiltonian description

Let

$$\Upsilon = J_1 X_1 + J_2 X_2 \tag{45}$$

be a dynamical system on the manifold $\mathcal{T}^*\mathcal{Q}$, with X_1 and X_2 obtained in (18) and (19). The relation (45) can be rewritten as

$$\Upsilon = \nu_a X^a + \nu_e X^e, \tag{46}$$

where

$$\nu_a = -2H_a,\, \nu_e = H_e, \ H_a = J_1 H_0, \ H_e = J_2 H_1, \ X^a = \frac{\partial}{\partial \Phi^a}, \ X^e = \frac{\partial}{\partial \Phi^e}, \tag{47}$$

$$\frac{\partial}{\partial \Phi^a} = \left(\frac{\partial}{\partial \varphi^1} + \frac{\partial}{\partial \varphi^2}\right), \quad \frac{\partial}{\partial \Phi^e} = -\left(\frac{\partial}{\partial \varphi^1} + \frac{\partial}{\partial \varphi^2}\right). \tag{48}$$

The vector fields X^a, X^e and the C^∞-functions H_a, H_e satisfy the following properties:

$$[X^i, X^j] = 0, \quad \mathcal{L}_{X^i} H_i = 0, \quad i, j \in \{a, e\}. \tag{49}$$

Let $\mathcal{N}$ be an open dense submanifold of $\mathcal{T}^*\mathcal{Q}$ on which Υ is explicitly integrable such that

$$X^a \wedge X^e \neq 0, \quad dH_a \wedge dH_e \neq 0. \tag{50}$$

Now, considering the coordinate system (H, Φ) with Φ^i, $i \in \{a, e\}$, which are closed differential 1-forms, the equations of motion of Υ are given by

$$\dot{\Phi}^a = -2H_a, \quad \dot{\Phi}^e = H_e, \quad \dot{H}_a = 0, \quad \dot{H}_e = 0, \tag{51}$$

with the functions H_a and H_e obeying condition (50). We can construct a closed 2-form, for $i \in \{a, e\}$,

$$\tilde{\omega} = \sum_i df^i(H_i) \wedge d\Phi^i, \tag{52}$$

which is non-degenerate as long as $df^a \wedge df^e \neq 0$, and

$$\iota_{X^i}\tilde{\omega} = -df^i, \quad \iota_\Upsilon \tilde{\omega} = -\sum_i \nu_i df^i, \quad \sum_i d\nu_i \wedge df^i = 0. \tag{53}$$

Notice that (53) is a necessary condition for $\iota_\Upsilon \tilde{\omega}$ to be exact, i.e., it is closed. Since $d\nu_a \wedge d\nu_e \neq 0$, the solutions of (53) are given by linear functions [17]

$$f^i = \sum_j L^{ij}\nu_j, \quad i,j \in \{a,e\}, \text{ where } L = \begin{pmatrix} -1/2 & 0 \\ 0 & 1 \end{pmatrix}. \tag{54}$$

Then, we get

$$f^a = -\frac{1}{2}\nu_a, \quad f^e = \nu_e. \tag{55}$$

From (47) and (55), we can rewrite (52) in the new coordinate system (ν, Φ) as

$$\tilde{\omega} = \sum_i df^i(\nu_i) \wedge d\Phi^i, \quad i \in \{a,e\}, \tag{56}$$

leading to the following form:

$$\tilde{\omega} = -\frac{1}{2} d\nu_a \wedge d\Phi^a + d\nu_e \wedge d\Phi^e. \tag{57}$$

The corresponding Hamiltonian description for Υ is given with the following quadratic Hamiltonian function

$$\tilde{H} = -\frac{1}{4}\nu_a^2 + \frac{1}{2}\nu_e^2. \tag{58}$$

In addition, from [34] other symplectic structures of the form (56) can be constructed, in which any f_i depending only on the corresponding frequency ν_i, $i \in \{a,e\}$, will be admissible as long as $\tilde{\omega}_b$, $b \in \{1,\ldots,n\}$, is non-degenerate, i.e., as long as $df^a \wedge df^e \neq 0$. From above, putting

$$f^a = \nu_a, \; f^e = \nu_e \; \text{ and } f^a = \nu_a^2, \; f^e = \nu_e^2, \tag{59}$$

we respectively obtain

$$\tilde{\omega}_1 = d\nu_a \wedge d\Phi^a + d\nu_e \wedge d\Phi^e \; \text{ and } \; \tilde{\omega}_2 = 2\nu_a d\nu_a \wedge d\Phi^a + 2\nu_e d\nu_e \wedge d\Phi^e. \tag{60}$$

Then, the $(1,1)$-tensor field $\mathcal{T} = \tilde{\omega}_2 \circ \tilde{\omega}_1^{-1}$ is constructed, taking the form

$$\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2, \tag{61}$$

where

$$\mathcal{T}_1 = 2\nu_a\left(\frac{\partial}{\partial \nu_a} \otimes d\nu_a + \frac{\partial}{\partial \Phi^a} \otimes d\Phi^a\right) \text{ and } \mathcal{T}_2 = 2\nu_e\left(\frac{\partial}{\partial \nu_e} \otimes d\nu_e + \frac{\partial}{\partial \Phi^e} \otimes d\Phi^e\right). \tag{62}$$

Finally, basing on [29–31], $\mathcal{T}_1$ and $\mathcal{T}_2$ are recursion operators for the dynamical system Υ. Hence, $\mathcal{T}$ is also a recursion operator for the dynamical system Υ as a sum of two recursion operators.

6. Quasi-bi-Hamiltonian structures

Basing on [8] and [9], in this part, we investigate the recursion operators for quasi-bi-Hamiltonian structures.

Definition 6. A Hamiltonian vector field Y on a symplectic manifold $(\mathcal{M}, \omega)$ is called quasi-bi-Hamiltonian if there exist another symplectic structure ω_1 and a nowhere-vanishing function g such that gY is a Hamiltonian vector field with respect to ω_1, i.e.,

$$\iota_Y \omega_0 = -dH_0, \quad \iota_{gY}\omega_1 = \iota_Y(g\omega_1) = -dH_1, \tag{63}$$

where H_0 and H_1 are integrals of motion for the Hamiltonian vector field Y; $g\omega_1$ is not closed in general.

A consequence of this definition is that the pair (ω_0, ω_1) determines a $(1,1)$-tensor field T defined as $T := \hat{\omega}_0^{-1} \circ \hat{\omega}_1$, that is, $\omega_0(Y, X) = \omega_1(TY, X)$, where X, Y are two Hamiltonian vector fields, and $\hat{\omega} := \iota_Y \omega$. In the action-angle coordinates (J, φ), the decomposition of the symplectic form $\omega' = \omega_1' + \omega_2'$, where

$$\omega_1' = dJ_1 \wedge d\varphi^1 - \left(\frac{2(J_1+J_2)^3}{m^2k^2\alpha} + 1\right) dJ_2 \wedge d\varphi^2, \tag{64}$$

$$\omega_2' = -dJ_1 \wedge d\varphi^2 + \left(\frac{2(J_1+J_2)^3}{m^2k^2\alpha} + 1\right) dJ_2 \wedge d\varphi^1, \tag{65}$$

shows that:

(i) ω_1' and ω_2' are not closed, i.e., $d\omega_1' \neq 0, d\omega_2' \neq 0$, where d is the exterior derivative. So, ω_1' and ω_2' are not symplectic.

(ii) $\iota_{X_H}\omega_1' = -dh_1'$, $\iota_{X_H}\omega_2' = -dh_2'$, where $h_1' = -h_2' = -\dfrac{2J_2}{m\alpha}$.

(iii) The functions h_1' and h_2' are first integrals of X_H, i.e., $X_H(h_1') = X_H(h_2') = 0$.

Proposition 7. *Hamiltonian vector field X_H is quasi-bi-Hamiltonian with respect to the two 2-forms (ω, ω_1'); idem for (ω, ω_2'). The weaker ω_i' recursion operators are given by:*

$$\widetilde{T}_1' := \omega^{-1} \circ \omega_1' \tag{66}$$

$$= \frac{\partial}{\partial J_1} \otimes dJ_1 + \frac{\partial}{\partial \varphi^1} \otimes d\varphi^1 - (2K+1)\left(\frac{\partial}{\partial J_2} \otimes dJ_2 + \frac{\partial}{\partial \varphi^2} \otimes d\varphi^2\right) \tag{67}$$

and

$$\widetilde{T}_2' := \omega^{-1} \circ \omega_2'$$

$$= (2K+1)\left(\frac{\partial}{\partial J_1} \otimes dJ_2 + \frac{\partial}{\partial \varphi^2} \otimes d\varphi^1\right) - \frac{\partial}{\partial \varphi^1} \otimes d\varphi^2 - \frac{\partial}{\partial J_2} \otimes dJ_1, \tag{68}$$

where $K = \dfrac{(J_1+J_2)^3}{m^2k^2\alpha}$ *and the 2-vector field* $\omega^{-1} = \dfrac{\partial}{\partial J_i} \wedge \dfrac{\partial}{\partial \varphi^i}$.

Similarly, ω'' can be re-expressed as the sum of two 2-forms as follows:

$$\omega'' = \omega_1'' + \omega_2'', \tag{69}$$

where

$$\omega_1'' = 2dJ_1 \wedge d\varphi^1 - \left(2J_2 + \frac{J_2(2J_2+1)}{(J_1+J_2)^2}\right) dJ_2 \wedge d\varphi^2, \tag{70}$$

$$\omega_2'' = -J_2 dJ_1 \wedge d\varphi^2 - 2\frac{(1+J_1)\varphi^1}{(J_1+J_2)} dJ_1 \wedge dJ_2 + \left(2 + \frac{2J_2+1}{(J_1+J_2)}\right) dJ_2 \wedge d\varphi^1$$
$$+ \left(\frac{2}{(J_1+J_2)} + \frac{2J_2+1}{(J_1+J_2)^2}\right)(J_2-1)\varphi^1 dJ_2 \wedge dJ_1. \tag{71}$$

As above:

(iv) ω_1'' and ω_2'' are not symplectic, i.e., $d\omega_1'' \neq 0, d\omega_2'' \neq 0$.

(v) $\iota_{X_H}\omega_1'' = -dh_1''$, $\iota_{X_H}\omega_2'' = -dh_2''$, where

$$h_1'' = \frac{k^2 m(3J_2(8J_2 - 6J_1 + 3) + J_1(2J_1+5))}{6(J_1+J_2)^3}, \tag{72}$$

$$h_2'' = \frac{k^2 m(J_2(-J_2 - 3J_1 + 12) + 8J_1)}{6(J_1+J_2)^3}. \tag{73}$$

(vi) h_1'' and h_2'' are also first integrals of X_H, i.e., $X_H(h_1'') = X_H(h_2'') = 0$.

Proposition 8. *Hamiltonian vector field X_H is quasi-bi-Hamiltonian with respect to the two 2-forms (ω, ω_1''); idem for (ω, ω_2''). The weaker ω_i'' recursion operators $\widetilde{T}_1''$ and $\widetilde{T}_2''$ are:*

$$\widetilde{T}_1'' := \omega^{-1} \circ \omega_1''$$
$$= 2\left(\frac{\partial}{\partial\varphi^1} \otimes d\varphi^1 + \frac{\partial}{\partial J_1} \otimes dJ_1\right) - J_2\left(2 + \frac{\widetilde{V}}{V^2}\right)\left(\frac{\partial}{\partial J_2} \otimes dJ_2 + \frac{\partial}{\partial\varphi^2} \otimes d\varphi^2\right), \tag{74}$$

$$\widetilde{T}_2'' := \omega^{-1} \circ \omega_2''$$
$$= \left(2 + \frac{\widetilde{V}}{V}\right)\left(\frac{\partial}{\partial J_1} \otimes dJ_2 + \frac{\partial}{\partial\varphi^2} \otimes d\varphi^1\right) - J_2\left(\frac{\partial}{\partial\varphi^1} \otimes d\varphi^2 + J_2\frac{\partial}{\partial J_2} \otimes dJ_1\right)$$
$$- \left(2 + \frac{\widetilde{V}}{V^2}(J_2-1)\right)\varphi^1\left(\frac{\partial}{\partial\varphi^1} \otimes dJ_2 - \frac{\partial}{\partial\varphi^2} \otimes dJ_1\right), \tag{75}$$

where $\widetilde{V} = 2J_2 + 1,\ V = J_1 + J_2$.

7. Concluding remarks

In this paper, we have constructed recursion operators for the Kepler dynamics in a deformed phase space by considering the equatorial orbit, computed the associated integrals of motion, and proved the existence of quasi-bi-Hamiltonian structures for the Kepler dynamics.

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Notes on integrable motion of two interacting curves and two-layer generalized Heisenberg ferromagnet equations

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Abstract. We study the integrable two-layer generalized Heisenberg ferromagnet equation (HFE). The relation between this generalized HFE and differential geometry of curves is established. Using this relation we found the geometrical equivalent counterpart of the two-layer spin system which is the two-component KdV equation. Finally, the gauge equivalence between these equations is established.

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1. Introduction

The Heisenberg ferromagnet equation (HFE)

$$\mathbf{A}_t = \mathbf{A} \wedge \mathbf{A}_{xx} \tag{1}$$

is one of basic fundamental nonlinear differential equations integrable by inverse scattering transform (IST) method [1]. The above $\mathbf{A} = (A_1, A_2, A\ )$ is a unit spin vector. As well-known, the HFE is geometrically and gauge equivalent to the nonlinear Schrodinger equation (NLSE) [2, 3]

$$iq_t + q_{xx} + 2|q|^2 q = 0. \tag{2}$$

The HFEs describe the nonlinear dynamics of one-layer ferromagnets [4–9]. At the same, it is well-known that ferromagnets have the multilayer nature. To describe such multilayer ferromagnets we need some type multilayer generalizations of the HFE (1). This is the first and main physical motivations of this paper. On the other hand, there are a lot of multicomponent generalizations of known integrable systems. For example, the NLSE (2) admits the following integrable two-component generalization:

$$\begin{aligned} iq_{1t} + q_{1xx} + 2(|q_1|^2 + |q_2|^2)q_1 = 0, \\ iq_{2t} + q_{2xx} + 2(|q_1|^2 + |q_2|^2)q_2 = 0, \end{aligned} \tag{3}$$

which is known as the Manakov system [10]. Such type integrable generalizations of soliton equations dictates also a need to construct integrable and nonintegrable multilayer extensions of the HFE (1). It is the second (perhaps mathematical) motivation for our study of multilayer generalizations of the HFE (1). Note that the one of two-layer spin systems was constructed in [11, 12] and reads as

$$\mathbf{A}_t = \mathbf{A} \wedge \mathbf{A}_{xx} + u_1\mathbf{A}_x + v_1\mathbf{H}_1 \wedge \mathbf{A},$$

$$\mathbf{B}_t = \mathbf{B} \wedge \mathbf{B}_{xx} + u_2\mathbf{B}_x + v_2\mathbf{H}_2 \wedge \mathbf{B},$$

where $\mathbf{B} = (B_1, B_2, B\ )$ is the second spin vector and u_j, v_j are some real functions of A_j, B_j. This set of equations (which known as the two-layer M-LIII equation) is the gauge and geometrical equivalent counterpart of the Manakov system (3) and hence is integrable.

The outline of the present paper is organized as follows. In Section 2, we present the two-layer M-IV equation. Also the relation between the motion of space curves and the two-layer M-IV equation is established. Then using this relation we found that the Lakshmanan (geometrical) equivalent counterpart of the M-LXXIII equation is the well-known two-component KdV equation. Section 3 is devoted to the integrable aspects of the considered systems. The gauge equivalence between the two-layer M-IV equation and the two-component KdV equation is established in Section 4. Some briefly information for the surface induced by the $SU(3)$ Γ-spin system is presented in Section 5. The paper is concluded by some comments in Section 6.

2. Integrable two-layer spin system

In this paper, we consider the following two-layer spin system called the two-layer M-IV equation

$$\begin{aligned} \mathbf{A}_t = \mathbf{A}_{xxx} + u_1\mathbf{A}_x + v_1\mathbf{A}, \\ \mathbf{B}_t = \mathbf{B}_{xxx} + u_2\mathbf{B}_x + v_2\mathbf{B}, \end{aligned} \tag{4}$$

where $\mathbf{A} = (A_1, A_2, A\ )$, $\mathbf{B} = (B_1, B_2, B\ )$ are unit spin vectors ($\mathbf{A}^2 = \mathbf{B}^2 = 1$) and

$$u_1 = \mathbf{A}_x^2 + 3\big(\sqrt{\mathbf{A}_x^2} + \sqrt{\mathbf{B}_x^2}\,\big), \quad v_1 = \frac{3}{2}\big(\mathbf{A}_x^2\big)_x,$$

$$u_2 = \mathbf{B}_x^2 + 3\big(\sqrt{\mathbf{A}_x^2} + \sqrt{\mathbf{B}_x^2}\,\big), \quad v_2 = \frac{3}{2}\big(\mathbf{B}_x^2\big)_x.$$

We will study this set of equations from the different points of view. We start from the construction the relation between the two-layer M-IV equation (4) and differential geometry of curves. Consider two interacting space curves in R^4 space. With each curves is related two set of unit vectors $\mathbf{e}_j$ and $\mathbf{l}_j$, respectively. They satisfy the Frenet–Serret equations. Before going further, here it is appropriate to make a few comments that is here some algebraic comments in order. At the level of Lie algebras we have that algebra $s\ (\ )$ matrices are just antisymmetric matrices $\ \times\ $. It turns out that the six-dimensional space of such $\ \times\ $ matrices decomposes into two three-dimensional subspaces that are each closed under taking commutators and each of them satisfies precisely the commutation relations of $s\ (3)$. So here we use just the well-known algebraic relation $s\ (\ ) = s\ (3) \times s\ (3)$ and/or for space $R^4 = R\ \ \times R\ $.

It is well-known that between some integrable (soliton) equations the so-called geometrical (Lakshmanan) equivalence takes place. In this section, our aim is to find the Lakshmanan equivalent counterpart of equation (4). To do that, let us return to the interacting two 3-dimensional curves in R^4. The motion of these curves is given by the following equations:

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e} \end{pmatrix}_x = C_1 \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e} \end{pmatrix}, \quad \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e} \end{pmatrix}_t = G_1 \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e} \end{pmatrix}, \tag{5}$$

$$\begin{pmatrix} \mathbf{l}_1 \\ \mathbf{l}_2 \\ \mathbf{l} \end{pmatrix}_x = C_2 \begin{pmatrix} \mathbf{l}_1 \\ \mathbf{l}_2 \\ \mathbf{l} \end{pmatrix}, \quad \begin{pmatrix} \mathbf{l}_1 \\ \mathbf{l}_2 \\ \mathbf{l} \end{pmatrix}_t = G_2 \begin{pmatrix} \mathbf{l}_1 \\ \mathbf{l}_2 \\ \mathbf{l} \end{pmatrix}. \tag{6}$$

Here $\mathbf{e}_j$ and $\mathbf{l}_j$ are the unit tangent ($j=1$), normal ($j=2$) and binormal ($j=3$) vectors to the curves, x is their common arclength parametrising the curves. The matrices C_j and G_j have the forms

$$C_1 = \begin{pmatrix} 0 & \kappa_1 & 0 \\ -\kappa_1 & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}, \qquad G_1 = \begin{pmatrix} 0 & \omega & -\omega_2 \\ -\omega & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 0 & \kappa_2 & 0 \\ -\kappa_2 & 0 & \tau_2 \\ 0 & -\tau_2 & 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 0 & \theta & -\theta_2 \\ -\theta & 0 & \theta_1 \\ \theta_2 & -\theta_1 & 0 \end{pmatrix}.$$

Curvatures and torsions of curves are given by the following formulas:

$$\kappa_1 = \sqrt{\mathbf{e}_{1x}^2}, \quad \tau_1 = \frac{\mathbf{e}_1 \cdot (\mathbf{e}_{1x} \wedge \mathbf{e}_{1xx})}{\mathbf{e}_{1x}^2},$$
$$\kappa_2 = \sqrt{\mathbf{l}_{1x}^2}, \quad \tau_2 = \frac{\mathbf{l}_1 \cdot (\mathbf{l}_{1x} \wedge \mathbf{l}_{1xx})}{\mathbf{l}_{1x}^2}.$$

The compatibility condition of equations (5)–(6) reads as

$$C_{1t} - G_{1x} + [C_1, G_1] = 0,$$
$$C_{2t} - G_{2x} + [C_2, G_2] = 0,$$

or in elements,

$$\begin{aligned} \kappa_{1t} = \omega_{\ x} + \tau_1\omega_2, \quad \tau_{1t} = \omega_{1x} - \kappa_1\omega_2, \\ \omega_{2x} = \tau\omega\ \ - \kappa\omega_1, \quad \kappa_{2t} = \theta_{\ x} + \tau_1\theta_2, \\ \tau_{2t} = \theta_{1x} - \kappa_1\theta_2, \quad \theta_{2x} = \tau_2\theta\ \ - \kappa\theta_1. \end{aligned} \tag{7}$$

Now we do the following identifications:

$$\mathbf{A} \equiv \mathbf{e}_1, \quad \mathbf{B} \equiv \mathbf{l}_1.$$

Then we have

$$\begin{aligned} \kappa_1^2 = \mathbf{A}_x^2, \quad \tau_1 = \frac{\mathbf{A} \cdot (\mathbf{A}_x \wedge \mathbf{A}_{xx})}{\mathbf{A}_x^2}, \\ \kappa_2^2 = \mathbf{B}_x^2, \quad \tau_2 = \frac{\mathbf{B} \cdot (\mathbf{B}_x \wedge \mathbf{B}_{xx})}{\mathbf{B}_x^2}. \end{aligned}$$

Now we want to simplify the problem. Namely, we assume that

$$\tau_1 = \tau_2 = 0.$$

This means we consider two plane curves without torsions. In this case we have

$$\omega_2 = \omega_1 = \theta_1 = \theta\ \ = 0.$$

At the same time from (7) we get

$$\kappa_{1t} = \omega_{\ x}, \quad \kappa_{2t} = \theta_{\ x}.$$

These results give us

$$\begin{aligned} \omega\ \ &= -\kappa_{1xx} - 3(\kappa_1 + \kappa_2)\kappa_1, \\ \theta\ \ &= -\kappa_{2xx} - 3(\kappa_1 + \kappa_2)\kappa_2. \end{aligned}$$

Finally we obtain the following set of equations:

$$\begin{aligned} \kappa_{1t} + \kappa_{1xxx} + 3[(\kappa_1 + \kappa_2)\kappa_1]_x = 0, \\ \kappa_{2t} + \kappa_{2xxx} + 3[(\kappa_1 + \kappa_2)\kappa_2]_x = 0. \end{aligned} \tag{8}$$

It is the desired set of equations and which is the Lakshmanan or geometrical equivalent counterpart of the two-layer M-IV equation (4). Integrability properties of the obtained set (8) and the two-layer M-IV equation (4) we will consider in the next sections.

3. Integrability of the two-layer M-IV equation

Our aim in this section is to study integrability aspects of the M-IV equation (4). To this end, let us consider the Lax representation of the form

$$\begin{aligned} \Psi_x = U\Psi, \\ \Psi_t = V\Psi, \end{aligned}$$

where

$$U = -i\lambda\Sigma + Q,$$
$$V = -\ i\lambda\ \Sigma + \ \lambda^2 Q + 2i\lambda F_1 + F_0.$$

Here

$$\Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & q_1 & q_2 \\ r_1 & 0 & 0 \\ r_2 & 0 & 0 \end{pmatrix}, \quad F_1 = \begin{pmatrix} -v & q_{1x} & q_{2x} \\ -r_{1x} & r_1 q_1 & r_1 q_2 \\ -r_{2x} & r_2 q_1 & r_2 q_2 \end{pmatrix},$$

$$F_0 = -Q_{xx} - 2v\Sigma Q - F_{02},$$

where $v = r_1 q_1 + r_2 q_2$ and

$$F_{02} = \begin{pmatrix} (r_{1x}q_1 - r_1 q_{1x}) + (r_{2x}q_2 - r_2 q_{2x}) & 0 & 0 \\ 0 & r_1 q_{1x} - r_{1x} q_1 & r_1 q_{2x} - r_{1x} q_2 \\ 0 & r_2 q_{1x} - r_{2x} q_1 & r_2 q_{2x} - r_{2x} q_2 \end{pmatrix}.$$

The compatibility condition

$$U_t - V_x + [U, V] = 0$$

gives the following set of equations:

$$\begin{aligned} q_{1t} + q_{1xxx} - 3r_1(q_1^2)_x - 3r_2(q_1 q_2)_x &= 0, \\ q_{2t} + q_{2xxx} - 3r_2(q_2^2)_x - 3r_1(q_1 q_2)_x &= 0, \\ r_{1t} + r_{1xxx} - 3q_1(r_1^2)_x - 3q_2(r_1 r_2)_x &= 0, \\ r_{2t} + r_{2xxx} - 3q_2(r_2^2)_x - 3q_1(r_1 r_2)_x &= 0. \end{aligned} \tag{9}$$

It can be considered as the two-component modified Korteweg–de Vries (KdV) equation. We now present some reductions of equations (9).

i) Let $r_1 = r_2 = -1$. In this case the set of equations (9) takes the form

$$\begin{aligned} q_{1t} + q_{1xxx} + 3[(q_1 + q_2)q_1]_x &= 0, \\ q_{2t} + q_{2xxx} + 3[(q_1 + q_2)q_2]_x &= 0. \end{aligned} \tag{10}$$

It is the two-component KdV equation, which is integrable as the exact reduction of the integrable set of equations (9).

ii) Now we consider the case $r_1 = \sigma_1 q_1,\ r_2 = \sigma_2 q_2$, where $\sigma_j = \pm 1$. For this reduction, the set (9) converted to the following set of equations

$$q_{1t} + q_{1xxx} - 6\sigma_1 q_1^2 q_{1x} - 3\sigma_2 q_2 (q_1 q_2)_x = 0,$$
$$q_{2t} + q_{2xxx} - 6\sigma_2 q_2^2 q_{2x} - 3\sigma_1 q_1 (q_1 q_2)_x = 0$$

It is nothing but the integrable two-component mKdV equation.

iii) Our third example is the following reduction: $r_1 = \sigma_1 \bar{q}_1,\ r_2 = \sigma_2 \bar{q}_2$. In this case we have

$$q_{1t} + q_{1xxx} - 6\sigma_1 \bar{q}_1 (q_1^2)_x - 3\sigma_2 \bar{q}_2 (q_1 q_2)_x = 0,$$
$$q_{2t} + q_{2xxx} - 6\sigma_2 \bar{q}_2 (q_2^2)_x - 3\sigma_1 \bar{q}_1 (q_1 q_2)_x = 0,$$

which is the well-known two-component complex mKdV equation.

iv) Our last example is following: $r_1 = \sigma_1 q_1,\ r_2 = \sigma_2 \bar{q}_2$, where we assume that q_1 is real function and q_2 is complex function. Then equations (9) take the form

$$q_{1t} + q_{1xxx} - 6\sigma_1 q_1^2 q_{1x} - 3\sigma_2 \bar{q}_2 (q_1 q_2)_x = 0,$$
$$q_{2t} + q_{2xxx} - 3\sigma_2 \bar{q}_2 (q_2^2)_x - 3\sigma_1 q_1 (q_1 q_2)_x = 0$$

It is some kind-integrable set of the mixed mKdV-complex mKdV equations.

4. Gauge equivalent equation

It is interesting to find the gauge equivalent equation to the set of equations (4) that is to the two-layer M-IV equation. With this aim, let us we consider the gauge transformation $\Phi = g^{-1}\Psi$, where $g = \Psi|_{\lambda=0}$. Then we have

$$\Phi_x = -i\lambda\Gamma\Phi, \tag{11}$$

$$\Phi_t = \left[-\ \ i\lambda\ \ \Gamma + 2\lambda^2\Gamma\Gamma_x + i\lambda\left(\Gamma_{xx} + \frac{3}{2}\Gamma\Gamma_x^2\right)\right]\Phi, \tag{12}$$

where

$$\Gamma = g^{-1}\Sigma g = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_1 \\ \Gamma_{21} & \Gamma_{22} & \Gamma_2 \\ \Gamma_{\ 1} & \Gamma_{\ 2} & \Gamma \end{pmatrix}.$$

The compatibility condition of equations (11)–(12) gives

$$\Gamma_t + \Gamma_{xxx} + \frac{3}{2}\left(\Gamma_x^2\Gamma\right)_x = 0. \tag{13}$$

This equation can be considered as the $su(3)$ form of the two-layer M-IV equation (4). From the definition of the Γ-matrix function follows that

$$\Gamma_x = g^{-1}\left[\Sigma, Q\right]g$$

so that

$$\Gamma_x^2 = -\ \begin{pmatrix} v & 0 & 0 \\ 0 & r_1 q_1 & r_1 q_2 \\ 0 & r_2 q_1 & r_2 q_2 \end{pmatrix}.$$

Hence we obtain

$$\mathrm{tr}\left(\Gamma_x^2\right) = -8v.$$

It is the Hamiltonian of equation (13):

$$H = \frac{1}{2}\int \mathrm{tr}\left(\Gamma_x^2\right). \tag{14}$$

Note that from (14) we can get the Hamiltonian of the two-layer M-IV equation (4) with the suitable reduction.

5. Integrable surface

Lastly let us briefly construct an integrable surface related with equation (13). We assume the following identification:

$$\Gamma \equiv R_x.$$

Then from (13) we have

$$R_t + R_{xxx} + \frac{3}{2}R_{xx}^2 R_x = 0.$$

Now we can define the first fundamental form of the surface as

$$I = \mathrm{tr}\left(R_t^2\right)dt^2 + 2\,\mathrm{tr}\left(R_t R_x\right)dxdt + \mathrm{tr}\left(R_x^2\right)dx^2.$$

Similarly we can construct the second fundamental form. These two forms allow us to construct the surface induced by the spin system (13) and which is integrable naturally.

6. Conclusions

In this paper, we have established the relation between the two-layer M-IV equation (4) and the two-component KdV equation (10). We have shown that the two-layer M-IV equation (4) and the two-component KdV equation (10) is the geometrically equivalent one to the other. Also the gauge equivalence between these equations is proved. Our results are significant for the deep understanding of integrable spin systems and their relations with differential geometry of curves and surfaces in multilayer and multicomponent case. Note that the two-component KdV equation and the two-component mKdV equation as well as the set of equations (9) can be viewed as one of members of the Manakov system hierarchy (3). At last we note that the construction of integrable class two and (in general n) interacting curves and surfaces is one of actual problems of modern differential geometry of curves and surfaces. Our work in this direction is in progress.

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About the solutions to the Witten–Dijkgraaf–Verlinde–Verlinde associativity equations and their Lie-algebraic and geometric properties

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Abstract. There is devised an algebraically feasible approach to investigating solutions to the oriented associativity equations, related with commutative and isoassociative algebras, interesting for applications in the quantum deformation theory and in some other fields of mathematics. The main construction is based on a modified version of the Adler–Kostant–Symes scheme, applied to the Lie algebra of the loop diffeomorphism group of a torus and modified for the case of the Gauss–Manin displacement equations, depending on a spectral parameter. Their interpretation as characteristic equations for some system of the Lax–Sato type vector field equations made it possible to derive the determining separated Hamiltonian evolution equations for the related structure matrices, generating commutative and isoassociative algebras under consideration.

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1. The introductory setting

As was mentioned in [6], the beauty of the theory of Frobenius manifolds is not only in multiple connections with other branches of mathematics, such as quantum cohomology, singularity theory and the theory of integrable systems. Even more amazing is that, some properties discovered in the study of particular classes of Frobenius manifolds often turn out to become universal structures of the theory thus proving to be important also for other classes of Frobenius manifolds. A crucial part of the Frobenius manifold theory is still in studying Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) type equations, first proposed in [4,21], and the related geometric structures, which have attracted considerable attention of specialists in topological quantum field theory and modern mathematical physics. In articles [15,17], generalizations of the WDVV equations, the so-called *oriented associativity equations,* there were proposed, describing both the displacement vector [5–8] relationships and isoassociative deformations [13] of associative algebras. Suitable reductions of the oriented associativity equations naturally arise in topological 2D-gravity [21, 22], singularity theory and complex geometry [12, 16], differential geometry of F-manifolds [14, 19] and theory of integrable systems [12, 18]. The important integrability and Hamiltonian properties of 2D-associativity equations were deeply studied in [3,10]. Being interested in the geometrical and Lie-algebraic properties of the oriented associativity equations, I was much inspired by [3, 19] and succeeded in their Hamiltonian reformulation within the modern Lie-algebraic Adler–Kostant–Symes approach as some integrable flows on the coadjoint space to a naturally constructed loop Lie algebra of the torus group diffeomorphisms. This made it possible to study the related Gauss–Manin displacement equations, depending on a spectral parameter. Their interpretation as characteristic equations for some system of the Lax–Sato type vector field equations made it possible to derive the determining separated Hamiltonian evolution equations for the related structure matrices, generating commutative and associative algebras under regard.

Recall that a Frobenius algebra is a pair $(A;\langle\cdot,\cdot\rangle)$, where A is a commutative associative algebra with a unity over a field $\mathbb{R}$, and $\langle\cdot,\cdot\rangle$ is a $\mathbb{R}$-bilinear symmetric nondegenerate invariant form on A, that is, $\langle x\cdot y, z\rangle = \langle x, y\cdot z\rangle$ for arbitrary vectors $x, y, z \quad A$.

Definition 1. Frobenius structure of the charge $d \quad \mathbb{Z}_+$ on the manifold M is a structure of a Frobenius algebra on the tangent spaces $T_t(M) = (A_t;\langle\cdot,\cdot\rangle_t)$, depending (smoothly, analytically, etc.) on the point $t \quad M$. It must satisfy the following axioms: *1*^{0} The metric $\langle\cdot,\cdot\rangle_t$ on M is flat. Denote by ∇ the Levi-Civita connection for the metric. The unity vector field e must be flat, $\nabla e = 0$; *2*^{0} Let c be the 3-tensor $c(x, y, z) := \langle x \cdot y, z\rangle_t$, $x, y, z \quad T_t(M)$. The 4-tensor $(\nabla_w c)(x, y, z)$ must be symmetric in $x, y, z, w \quad T_t(M)$; *3*^{0} A linear vector field $E \quad Vect(M)$ must be fixed on M, i.e., $\nabla\nabla E = 0$, such that $[E, x \cdot y] - [E, x] \cdot y - x \cdot [E, y] = x \cdot yE\langle x, y\rangle_t - \langle [E, x], y\rangle_t - \langle x, [E, y]\rangle_t = (2 - d)\langle x, y\rangle_t$.

The last condition means that the derivations E and $Q_{\Gamma(M)} := id + \mathrm{ad}_E$ define on the space $\Gamma(M)$ of vector fields on M a structure of graded Frobenius algebra over the graded ring of functions on M. Flatness of the metric $\langle\cdot,\cdot\rangle_t$ implies *local existence* of a system of *flat coordinates* $t := (t_0, \ldots, t_n) \quad \mathbb{R}^{n+1}$ on M. We will denote $\{\eta_{ij}(t) : i, j = \overline{0, n}\}$ the constant Gram matrix of the metric in these coordinates $\eta_{ij}(t) := \left\langle \frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_j} \right\rangle_t$ for all $i, j = \overline{0, n}, t \quad M$. The inverse matrix $\{\eta^{ij}(t) : i, j = \overline{0, n}\}$ defines the inner product on the cotangent planes as $\left\langle dt_i, dt_j \right\rangle_t = \eta^{ij}(t), i, j = \overline{0, n}, t \quad M$. The flat coordinates will be chosen in such a way that the unity $e \quad A$ of the Frobenius algebra coincides with $\partial/\partial t_0, e = \partial/\partial t_0$. In these flat coordinates on M one can introduce the structure constants of the Frobenius algebra $A_t = T_t(M)$ at $t \quad M : \frac{\partial}{\partial t_i} \cdot \frac{\partial}{\partial t_j} = \sum\limits_{k=\overline{0,n}} C_{ij}^k \frac{\partial}{\partial t_k}$ at for all $i, j = \overline{0,n}$, which can be locally represented by the third derivatives of a function $F(t)$ as $C_{ij}^k(t) = \eta^{ks}(t) \frac{\partial\ F(t)}{\partial t_s \partial t_i \partial t_j}$, $\eta_{ij}(t) = \frac{\partial\ F(t)}{\partial t_0 \partial t_i \partial t_j}$ for any $i, j, k = \overline{0,n}$. The function $F(t), t \quad M$, is called *potential* of the Frobenius manifold. It satisfies the following system of *WDVV associativity equations*:

$$\sum_{k,s=\overline{0,n}} \frac{\partial\ F(t)}{\partial t_i \partial t_j \partial t_k} \eta^{ks}(t) \frac{\partial\ F(t)}{\partial t_s \partial t_p \partial t_q} = \sum_{k,s=\overline{0,n}} \frac{\partial\ F(t)}{\partial t_i \partial t_j \partial t_k} \eta^{ks}(t) \frac{\partial\ F(t)}{\partial t_s \partial t_j \partial t_i}, \tag{1}$$

suggested independently by Witten in 1990 and Dijgraaf–Verlinde–Verlinde in 1991. In 1992 B. Dubrovin deeply analyzed these Frobenius manifold structures and reformulated them in more clear form. The vector field E is called Euler vector field. In the *flat coordinates* it must have the form $E = \sum_{i=\overline{0,n}} (a^{ij} t_j + b^i) \frac{\partial}{\partial t_i}$ where $a^{0j} = 1, j = \overline{0,n}, b^0 = 0$. Moreover, vanishing of the curvature of the connection ∇ is essentially equivalent to the axioms of Frobenius manifold. Introducing a tangent vector $x := (x_j : j = \overline{0,n}) \quad T(M)$, the flatness condition can be rewritten as the system of compatible linear equations

$$\partial x(t;\lambda)/\partial t_k = \lambda^{-1} C_k(t) x(t;\lambda), \tag{2}$$

for $k = \overline{0,n}$ and arbitrary parameter $\lambda \quad \mathbb{C}\backslash\{0\}$, being augmented with the Euler vector compatibility equation $\partial x(t;\lambda)/\partial\lambda = (U(t) + \lambda V) x(t;\lambda)$, where matrices $C_i(t) := \{C_{ij}^k(t) : j,k = \overline{0,n}\}, i = \overline{0,n}$, $U(t) := \{\sum_{s=\overline{0,n}} E^s C_{sj}^i(t) : i,j = \overline{0,n}\}$ and $V(t) := \frac{(2-d)}{2} - \nabla E$ is an antisymmetric operator on the tangent bundle $T(M)$ with respect to $\langle\cdot,\cdot\rangle_t$, that is, $\langle Vx, y\rangle_t + \langle x, Vy\rangle_t = 0$ for any vector $x, y \quad T(M)$.

As was pointed out by B. Dubrovin [5], because of vanishing of the torsion and curvature of the connection ∇, there locally exist on the manifold $M \otimes \mathbb{C}$ $n+1$ independent new flat functions $\{t_j(t;\lambda) \quad \mathbb{R} : j = \overline{0,n}\}$, called *deformed flat coordinates* on a Frobenius manifold. The analytic properties of deformed flat coordinates as multi-valued functions of $\lambda \quad \mathbb{C}\backslash\{0\}$ can be used, in particular, for describing moduli of semisimple Frobenius manifolds. In our work we are mainly interested in studying geometric structures, related with these deformed flat coordinates on a Frobenius manifold, generating the diffeomorphism group of the complexified torus $\mathbb{T}_{\mathbb{C}}^{n+1}$ and the related oriented associativity equations for the structure matrices $C_i(t) \quad \mathrm{End}\ \mathbb{E}^{n+1}, t \quad \mathbb{R}^n, i = \overline{0,n}$, endowing the symmetry constraints $C_{kj}^i(t) = C_{jk}^i(t)$ for all $i, j, k = \overline{0,n}$, are given in case of a commutative Frobenius algebra by the following relationships:

$$[C_m, C_k] = 0, \quad \frac{\partial C_k}{\partial t_m} = \frac{\partial C_m}{\partial t_k}, \tag{3}$$

which are satisfied for all $k,m=\overline{0,n}$. The first condition of (3) means that the commutative algebra in question is associative and the second one of (3) means that there is considered its invariant isoassociative [13] quantum deformation.

As it was already mentioned above a crucial observation was done in [5] and later in [13, 15], the oriented associativity equations (3) can be rewritten equivalently as the compatibility condition of the Gauss–Manin type [15] linear equations (2), where the vector $x(t,\lambda)\quad T_t(M)$ can be interpreted as a set of complexified diffeomorphisms of the torus $x(\cdot;\lambda):\mathbb{T}^{n+1}_{\mathbb{C}}\to\mathbb{T}^{n+1}_{\mathbb{C}}$, parametrically depending both on the parameter $\lambda\quad\mathbb{C}\backslash\{0\}$ and the temporal evolution variable t $\quad\mathbb{R}^{n+1}$. The latter makes it possible to construct a new set of Lax–Sato type vector field equations

$$A^{(k)}:=\frac{\partial}{\partial t_k}+\lambda^{-1}\sum_{j=\overline{0,n}}\Big(\sum_{s=\overline{0,n}}C^s{}_{kj}(t)x_s\Big)\frac{\partial}{\partial x_j}, \tag{4}$$

compatible for all $t_k\quad\mathbb{R}$, $k=\overline{0,n}$ and $\lambda\quad\mathbb{C}\backslash\{0\}$. This property can be formulated as the following proposition.

Proposition 2. *The set of oriented associativity equations* (3) *is equivalent to a set of the compatible for all* $k,m=\overline{0,n}$ *commutator Lax–Sato type vector field relationships*

$$\Big[A^{(k)},A^{(m)}\Big]=0 \tag{5}$$

for the vector fields (4).

As a very interesting example of the construction above one can obtain for the special case $n=2$, taking into account a reduction of the commuting matrices $C_j\quad$ End $\mathbb{E}\ $, $j=\overline{0,2}$, presented in [3, 5, 6]. Namely, assume that the smooth mapping $F:\mathbb{R}\ \to\mathbb{R}$ is representable as

$$F(t)=\frac{1}{2}t_0^2t_2+\frac{1}{2}t_0t_1^2+f(t_0,t_1,t_2), \tag{6}$$

where a smooth mapping $f:\mathbb{R}^2\to\mathbb{R}$ satisfies, as follows from (1) in the form $(\partial/\partial t_1\circ\partial/\partial t_1)\circ\partial/\partial t_2=\partial/\partial t_1\circ(\partial/\partial t_1\circ\partial/\partial t_2)$, $\partial/\partial t_0\circ\partial/\partial t_j=\partial/\partial t_j, j=\overline{0,2}$, such a partial differential equation:

$$f^2_{t_1t_1t_2}-f_{t_2t_2t_2}-f_{t_1t_1t_1}f_{t_1t_2t_2}=0 \tag{7}$$

for any $(t_0,t_1,t_2)\quad\mathbb{R}\ $. Equation (7), as it follows from (3), was shown by B. Dubrovin and later used by Yu. Manin [5, 6, 15, 16], is representable as the system of compatible linear differential equations for any $\lambda\quad\mathbb{R}\backslash\{0\}$:

$$\frac{\partial x}{\partial t_0}=\frac{1}{\lambda}\,C_0x,\quad \frac{\partial x}{\partial t_1}=\frac{1}{\lambda}\,C_1x,\quad \frac{\partial x}{\partial t_2}=\frac{1}{\lambda}\,C_2x \tag{8}$$

on points $x\quad\mathbb{E}\ $, determined by matrices

$$C_0=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad C_1=\begin{pmatrix}0&1&0\\b&a&1\\c&b&0\end{pmatrix},\quad C_2=\begin{pmatrix}0&0&1\\c&b&0\\b^2-ac&c&0\end{pmatrix}, \tag{9}$$

where $a := f_{t_1t_1t_1}$, $b := f_{t_1t_1t_2}$, $c := f_{t_1t_2t_2}$ and generating the corresponding loop $\mathrm{Diff}(\mathbb{R}\ )$-group diffeomorphisms. It is easy also to check that matrices (9) satisfy the determining matrix equations (3):

$$
\begin{aligned}
&[C_1, C_2] = 0, \\
&\frac{\partial C_1}{\partial t_2} = \frac{\partial C_2}{\partial t_1}, \quad \frac{\partial C_j}{\partial t_0} = 0
\end{aligned}
\tag{10}
$$

for any $t_j \quad \mathbb{R}, j = \overline{0,2}$.

Remark 3. It is worth to mention here that these commutator relationships are equivalent to the matrix relationships (3), where the matrices $C_k(t) \quad \mathrm{End}\, \mathbb{E}^{n+1}$, $k = \overline{0,n}, t \quad \mathbb{R}^{n+1}$, in general, cannot satisfy the symmetry property $C^i_{kj}(t) = C^i_{jk}(t)$ for all $i, j, k = \overline{0,n}$, $t \quad \mathbb{R}^{n+1}$, and which should be imposed on the vector fields (4) separately.

The commutator representations (5) make it possible to devise a Lie-algebraic and related geometric descriptions of the WDVV type equations, an attempt which is presented below.

2. The Lie-algebraic integrability analysis

Let $\tilde{\mathrm{G}}_\pm := \widetilde{\mathrm{Diff}}_\pm(\mathbb{T}^{n+1})$, $n \quad \mathbb{Z}_+$, be subgroups of the loop diffeomorphisms group $\widetilde{\mathrm{Diff}}(\mathbb{T}^{n+1}) := \{\mathbb{C} \supset \mathbb{S}^1 \to \mathrm{Diff}(\mathbb{T}^{n+1})\}$, holomorphically extended in the interior $\mathbb{D}^1_+ \subset \mathbb{C}$ and in the exterior $\mathbb{D}^1_- \subset \mathbb{C}$ regions of the unit disc $\mathbb{D}^1 \subset \mathbb{C}^1$, such that for any $g(\lambda) \quad \tilde{G}_-$, $\lambda \quad \mathbb{D}^1_-$, $g(\infty) = 1 \quad \mathrm{Diff}(\mathbb{T}^n)$. The corresponding Lie subalgebras $\tilde{\mathcal{G}}_\pm := \widetilde{\mathrm{diff}}_\pm(\mathbb{T}^n)$ of the loop subgroups $\tilde{G}_\pm$ are vector fields on $\mathbb{T}^{n+1}$ holomorphic, respectively, on $\mathbb{D}^1_\pm \subset \mathbb{C}^1$, where for any $\tilde{a}(\lambda) \quad \tilde{\mathcal{G}}_-$ the value $\tilde{a}(\infty) = 0$. The split loop Lie algebra $\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_+ \oplus \tilde{\mathcal{G}}_-$ can be naturally identified with a dense subspace of the dual space $\tilde{\mathcal{G}}^*$ through the pairing $(\tilde{l}, \tilde{a}) := \int_{\mathbb{T}^n} \mathrm{res}\, \langle \tilde{l}(x;\lambda),\ \tilde{a}(x;\lambda)\rangle d^n x$, where $\langle \cdot, \cdot \rangle$ is the convolution on the product $\Gamma(T^*(\mathbb{T}^{n+1})) \times \Gamma(T(\mathbb{T}^{n+1}))$ of differential forms, $\Gamma(T^*(\mathbb{T}^{n+1})) \simeq \tilde{\mathcal{G}}^*$ and vector fields $\Gamma(T(\mathbb{T}^{n+1})) \simeq \tilde{\mathcal{G}}$ on torus $\mathbb{T}^{n+1}$ with representatives $\tilde{l}(x;\lambda) := \sum_{k=\overline{0,n}} l^{(k)}(x;\lambda) dx_k \quad \tilde{\mathcal{G}}^*$, $\tilde{a}(x;\lambda) := \sum_{k=\overline{0,n}} a^{(k)}(x;\lambda)\partial/\partial x_k \quad \tilde{\mathcal{G}}$, respectively. For the gradient $\nabla h(\tilde{l}) \quad \tilde{\mathcal{G}}$ at a point $\tilde{l} \quad \tilde{\mathcal{G}}^*$ one can determine its projections $\nabla h(\tilde{l})_\pm \quad \tilde{\mathcal{G}}_\pm$ on the subalgebras $\tilde{\mathcal{G}}_\pm$, respectively, satisfying the splitting $\nabla h(\tilde{l}) = \nabla h(\tilde{l})_+ \oplus \nabla h(\tilde{l})$. If now to take a functional $h \quad \mathrm{I}(\tilde{\mathcal{G}}^*) \subset \mathrm{D}(\tilde{\mathcal{G}}^*)$ to be a Casimir one, satisfying the condition $\mathrm{ad}^*_{\nabla h(\tilde{l})}\tilde{l} = 0$ at any $\tilde{l} \quad \tilde{\mathcal{G}}^*$, it is easy to check that the subspaces $\tilde{\mathcal{G}}^*_+$ and $\tilde{\mathcal{G}}^*_-$ are invariant with respect to the co-adjoint action of the Lie subalgebras $\tilde{\mathcal{G}}_+$ and $\tilde{\mathcal{G}}_-$ on element $\tilde{l} \quad \tilde{\mathcal{G}}^*$, respectively. This property is crucial for constructing commuting to each other Hamiltonian flows on the adjoint space $\tilde{\mathcal{G}}^*$ within the mentioned above Adler–Kostant–Symes scheme. Namely, let $h^{(t_k)} \quad \mathrm{I}(\tilde{\mathcal{G}}^*), k = \overline{0,n}$, be functionally independent Casimir invariants on $\tilde{\mathcal{G}}^*$ and construct the following Hamiltonian flows with respect to the classical [1,2,9,11,20] Lie-Poisson structure on $\tilde{\mathcal{G}}^*$ as

$$\partial\tilde{l}/\partial t_k = -\operatorname{ad}^*_{\nabla h^{(t_k)}(\tilde{l})_-}\tilde{l} \tag{11}$$

with respect to evolution variables $t_k \quad \mathbb{R}, k = \overline{0,n}$. As the functionals $h^{(t_k)} \quad \mathrm{I}(\tilde{\mathcal{G}}^*), k = \overline{0,n}$, are chosen to be Casimir ones, the Hamiltonian flows (11) prove to be commuting to each other. The latter property, owing to the expressions (11), can be equivalently rewritten as the compatibility conditions

$$\left[\frac{\partial}{\partial t_k} + \nabla h^{(t_k)}(\tilde{l})_-, \frac{\partial}{\partial t_m} + \nabla h^{(t_m)}(\tilde{l})_-\right] = 0 \quad \text{for all } k, m = \overline{0,n}. \tag{12}$$

Now it is easy to observe that the commutator relationships (5) can be identified exactly with those (12), if to assume that there exists such an element $\tilde{l} \quad \tilde{\mathcal{G}}^*$, for which $A^{(k)} = \frac{\partial}{\partial t_k} + \nabla h^{(t_k)}(\tilde{l})_-$ for every $k \ = \overline{0,n}$. From (15) one derives [11,20] that

$$\nabla h^{(t_k)}(\tilde{l})_- = \lambda^{-1}\sum_{j=\overline{0,n}}\left(\sum_{s=\overline{0,n}} C^s{}_{kj}(t)x_s\right)\frac{\partial}{\partial x_j} \tag{13}$$

for every $k \ = \overline{0,n}$. Thus, we have reduced the problem of describing a set of the structure matrices $C_k(t) := \left\{C^s_{kj}(t) = C^s_{jk}(t) : j, s = \overline{0,n}\right\} \quad \operatorname{End}\mathbb{E}^{n+1}, k = \overline{0,n}$, for any $t \quad \mathbb{R}^{n+1}$ to that of describing the corresponding independent Casimir invariants $h^{(t_k)} \quad \mathrm{I}(\tilde{\mathcal{G}}^*), k = \overline{0,n}$, whose suitably projected gradients $\nabla h^{(t_k)}(\tilde{l})_- \quad \tilde{\mathcal{G}}_-, k = \overline{0,n}$, at some seed element $\tilde{l} \quad \tilde{\mathcal{G}}^*$ coincide exactly with the expressions (13).

To solve this problem, we need to start from the determining equation, namely $\operatorname{ad}^*_{\nabla h(\tilde{l})}\tilde{l} = 0$ at any $\tilde{l} \quad \tilde{\mathcal{G}}^*$ for a Casimir invariant functional $h \quad \mathrm{I}(\tilde{\mathcal{G}}^*)$ in the following componentwise form:

$$\langle\partial/\partial x, \circ\nabla h(l)\rangle l + \left\langle l, \frac{\partial}{\partial x}\nabla h(l)\right\rangle = 0, \tag{14}$$

where we put, by definition, $\nabla h(\tilde{l}) := \langle\nabla h(l), \partial/\partial x\rangle$, $\tilde{l} := \langle l, dx\rangle$. Having taken into account the expressions (13), we need to find exactly $n \quad \mathbb{Z}_+$ solutions to the equation (14) at some seed element $\tilde{l} \quad \tilde{\mathcal{G}}^*$, for which the following relationships $\nabla h^{(t_k)}(\tilde{l})_- = \left\langle\nabla h^{(t_k)}(l)_-, \partial/\partial x\right\rangle$, where the vector

$$\nabla h^{(t_k)}(l)_- = \left\{\lambda^{-1}\sum_{s=\overline{0,n}} C^s_{kj}(t)x_s : j = \overline{0,n}\right\}, \tag{15}$$

hold for all $x \quad \mathbb{T}^{n+1}$, $k = \overline{0,n}$, and any parameter $\lambda \quad \mathbb{C}\backslash\{0\}$. Based on the relationships (14) one can easily write down general asymptotic as $\lambda \to 0$ gradient expressions $\nabla h^{(t_k)}(\tilde{l}) \quad \tilde{\mathcal{G}}$,

$$\nabla h^{(t_k)}(\tilde{l}) = \left\langle\nabla h^{(t_k)}(l), \partial/\partial x\right\rangle, \quad \nabla h^{(t_k)}(l) \sim \sum_{j\in\mathbb{Z}_+}\lambda^{j-1}\varphi^{(k)}_j(l), \tag{16}$$

where $\varphi_0^{(k)}(l) = \big\{ \sum_{s=\overline{0,n}} C^s_{kj}(t)x_s : j = \overline{0,n} \big\}$ for all $k = \overline{0,n}$. Taking into account the expressions (15), a seed element $\tilde{l} \quad \tilde{\mathcal{G}}^*$ can be represented in the form $\tilde{l} = \langle l, dx\rangle,\ l = l_0 + \lambda^{-1} l_{-1}$, *a priori* generating asymptotic as $\lambda \to 0$ solutions (16) of the determining equation (14). Really, substituting expansions (16) into (14), one obtains that

$$\begin{aligned}
&\Big\langle \frac{\partial}{\partial x}, \circ\varphi_0^{(k)} \Big\rangle l_{-1} + \Big\langle \varphi_0^{(k)}, \frac{\partial}{\partial x} \Big\rangle l_{-1} + \Big\langle l_{-1}, \Big(\frac{\partial}{\partial x}\varphi_0^{(k)}\Big) \Big\rangle = 0, \\
&\Big\langle \frac{\partial}{\partial x}, \circ\varphi_0^{(k)} \Big\rangle l_0 + \Big\langle \frac{\partial}{\partial x}, \circ\varphi_1^{(k)} \Big\rangle l_{-1} + \Big\langle \varphi_1^{(k)}, \frac{\partial}{\partial x} \Big\rangle l_{-1} \\
&+ \Big\langle \varphi_0^{(k)}, \frac{\partial}{\partial x} \Big\rangle l_0 + \Big\langle l_0, \Big(\frac{\partial}{\partial x}\varphi_0^{(k)}\Big) \Big\rangle + \Big\langle l_{-1}, \Big(\frac{\partial}{\partial x}\varphi_1^{(k)}\Big) \Big\rangle = 0, \ldots,
\end{aligned} \tag{17}$$

and so on for any $k = \overline{0,n}$. As an example, one can take a simplest linear in $x \quad \mathbb{T}^{n+1}$ solution to the first equation of (17) and given by the vector expression $l_{-1} = \sum_{s=\overline{0,n}} \bar{l}^s_{-1}(t)x_s$, where the matrix $\bar{l}_{-1} := \big\{\bar{l}^{(s)}_{-1,j}(t) : j, s = \overline{0,n}\big\}$ satisfies the matrix equation $(\operatorname{tr} C_k)\bar{l}_{-1} + C_k^{\intercal}\bar{l}_{-1} + \bar{l}_{-1}C_k = 0$ for any $k = \overline{0,n}$. The latter means that if $\operatorname{tr} C_k \neq 0$, $k = \overline{0,n}$, we need to take such a matrix $\bar{l}_{-1} \quad \operatorname{End}\mathbb{E}^{n+1}$, for which the matrix equation $\bar{l}_{-1} + \bar{l}_{-1}\bar{C}_j + \bar{C}_j^{\intercal}\bar{l}_{-1} = 0$ possesses exactly $n \quad \mathbb{Z}_+$ linearly independent solutions $\big\{\bar{C}_k \quad \operatorname{End}\mathbb{E}^{n+1} : k = \overline{0,n}\big\}$, determining the matrices $C_k \quad \operatorname{End}\mathbb{E}^{n+1}$ as $C_k = \xi_k\overline{C}_k$ for arbitrary $\xi_k \quad \mathbb{R}\backslash\{0\}$ and every $k = \overline{0,n}$. Whence we can derive that the resulting Casimir functionals $h^{(t_k)} \quad \mathrm{I}(\tilde{\mathcal{G}}^*), k = \overline{0,n}$, are functionally independent and generate independent and commuting Hamiltonian flows (11). The case $\operatorname{tr} C_k = 0$ for all $k = \overline{0,n}$ results in the equation $\bar{l}_{-1}C_k + C_k^{\intercal}\bar{l}_{-1} = 0$,which for a suitably chosen matrix $\bar{l}_{-1} \quad \operatorname{End}\mathbb{E}^{n+1}$ also provides $n \quad \mathbb{Z}_+$ linearly independent solutions $\big\{C_k \quad \operatorname{End}\mathbb{E}^{n+1} : k = \overline{0,n}\big\}$and where the sign "$\intercal$" means the usual matrix transposition. The general case $\operatorname{tr} C_k \neq 0$ for all $k = \overline{1,p}$ and $\operatorname{tr} C_k = 0$ for all $k = \overline{p+0,n}$ and some $p \quad \{0, 2, \ldots, n\}$ reduces simply to finding such a matrix $\bar{l}_{-1} \quad \operatorname{End}\mathbb{E}^{n+1}$, for which the matrix system

$$\begin{aligned}
\bar{l}_{-1} + \bar{l}_{-1}\bar{C}_j + \bar{C}_j^{\intercal}\bar{l}_{-1} = 0 \quad &\text{for } j = \overline{1,p}, \\
\bar{l}_{-1}\bar{C}_j + \bar{C}_j^{\intercal}\bar{l}_{-1} = 0 \quad &\text{for } j = \overline{p+0,n}
\end{aligned} \tag{18}$$

possesses exactly $p \quad \mathbb{Z}_+$ linearly independent solutions $\big\{\bar{C}_k \quad \operatorname{End}\mathbb{E}^{n+1} : k = \overline{1,p}\big\}$ and $(n-p) \quad \mathbb{Z}_+$ independent solutions $\big\{C_k \quad \operatorname{End}\mathbb{E}^{n+1} : k = \overline{p+0,n}\big\}$, respectively. Based on the obtained this way matrix $\bar{l}_{-1} \quad \operatorname{End}\mathbb{E}^{n+1}$ one can successively proceed to finding a next matrix element $l_0 \quad \operatorname{End}\mathbb{E}^{n+1}$, solving the second equation of (17), depending on the unknown, yet arbitrary, elements $\varphi_1^{(k)}, k = \overline{0,n}$. For the vector $l_0 \quad \operatorname{End}\mathbb{E}^{n+1}$ to be obtained analytically, there to be taken into account compatibility conditions stemming from the second equation of (17). Before analyzing this and related problems, let us formulate the results above as the next proposition.

Proposition 4. *The linear in* $x \quad \mathbb{T}^{n+1}$ *solution* $l_{-1} = \sum_{s=\overline{0,n}} \bar{l}^{s}_{-1}(t)x_s \quad \mathbb{E}^{n+1}$ *to the first determining equation of* (17) *is, in general, generated by a matrix* $\bar{l}_{-1}$ $\operatorname{End}\mathbb{E}^{n+1}$, *such that the matrix system* (18) *for some* $p = \overline{0,n}$ *possesses exactly* $p \quad \{0,1,2,\dots,n\}$ *linearly independent solutions* $\{\bar{C}_k \quad \operatorname{End}\mathbb{E}^{n+1} : k = \overline{0,p}\}$ *and* $(n-p) \quad \{0,1,2,\dots,n\}$ *independent solutions* $\{C_k \quad \operatorname{End}\mathbb{E}^{n+1} : k = \overline{p+0,n}\}$, *respectively.*

The preceding, linear in $x \quad \mathbb{T}^{n+1}$ solution, $l_{-1} = \sum_{s=\overline{0,n}} \bar{l}^{s}_{-1}(t)x_s \quad \mathbb{E}^{n+1}$ does not, evidently, cover all possible solutions to the determining equations (17), exactly depending on the structural matrices $C_k(t) \quad \operatorname{End}\mathbb{E}^{n+1}$, $k=\overline{0,n}$, and $t \quad \mathbb{R}^{n+1}$. To analyze the general case, let us rewrite, preliminarily, the first equation of (17) as

$$\sum_{s=\overline{0,n}} (C_k x)_s \frac{\partial l_{-1,j}}{\partial x_s} + \sum_{s=\overline{0,n}} C^j_{ks} l_{-1,s} + \Big(\sum_{s=\overline{0,n}} C^s_{ks}\Big) l_{-1,j} = 0, \tag{19}$$

where $j = \overline{0,n}$ and a vector $l_{-1} := \{l_{-1,j}(x) \quad \mathbb{R} : s = \overline{0,n}\} \quad \mathbb{E}^{n+1}$. We have obtained a set of linear nonuniform vector field equations (19) with coefficients depending both on the vector $x \quad \mathbb{T}^{n+1}$ and on the structure matrices $C_k(t) \quad \operatorname{End}\mathbb{E}^{n+1}, k=\overline{0,n}$. The necessary condition for the solvability of the vector field equations (19) consists in existence of a common set of first integrals for the system of characteristic equations $\frac{\partial x_j}{\partial \tau_k} = \sum_{s=\overline{0,n}} C^s_{kj} x_s$with respect to evolution parameters $\tau_k \quad \mathbb{R}$ for all $k,j=\overline{0,n}$. Based on the corresponding compatible set of the first integrals for $\gamma_j(x; C_1, C_2, \dots, C_n), j = \overline{1,n}$, one can successfully solve the determining nonuniform vector fields equations (19) for the searched vector $l_{-1} \quad \mathbb{E}^{n+1}$, if to take additionally into account the next set of compatible linear characteristic equations

$$\frac{\partial l_{-1}}{\partial \tau_k} + C_k^{\intercal}\bar{l}_{-1} + \bar{l}_{-1}C_k = 0 \tag{20}$$

for all $k = \overline{0,n}$. The compatible system (20) also generates a common set of first integrals $\mu_j(l^{(1)}_{-1}, l^{(2)}_{-1}, \dots, l^{(n)}_{-1}), j = \overline{1,n}$, which make it possible to construct general solutions to the vector field equations (19) in the form

$$\Phi^{(k)}(\gamma_1,\gamma_2,\dots,\gamma_n;\mu_1,\mu_2,\dots,\mu_n) = 0,$$

where smooth mappings $\Phi^{(k)} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, k = \overline{1,n}$, are such that

$$\det\left(\frac{\partial \Phi^{(k)}}{\partial \mu_j}, \frac{\partial \Phi^{(k)}}{\partial \gamma_s}\right)_{j,s=\overline{1,n}} \neq 0$$

for any $k = \overline{1,n}$. Returning now back to the component $l_0 \quad \mathbb{E}^{n+1}$, we need to mention, taking into account that the expansion (16) is *a priori* infinite with the nonzero vector components $\varphi^{(k)}_j \quad \mathbb{E}^{n+1}$, $j \quad \mathbb{Z}_+$, for one $k = \overline{0,n}$, it could be chosen as an arbitrary *nonzero* functional vector, allowing to calculate successively the mentioned above infinite hierarchy of vector components from equations (17).

Now, based on the determined this way seed element $\tilde{l} = \langle l_0 + \lambda^{-1} l_{-1}, dx \rangle \quad \tilde{\mathcal{G}}^*$, one can construct the related evolution flows (11) in the following equivalent vector form:

$$\begin{aligned} \frac{\partial}{\partial t_j}\big(l_0 + \lambda^{-1} l_{-1}\big) = \left\langle \frac{\partial}{\partial x}, \circ \nabla h^{(t_k)}(l)_- \right\rangle \big(l_0 + \lambda^{-1} l_{-1}\big) \\ + \left\langle \big(l_0 + \lambda^{-1} l_{-1}\big), \frac{\partial}{\partial x}\big(\nabla h^{(t_k)}(l)_-\big) \right\rangle, \end{aligned} \tag{21}$$

which is *a priori* compatible for any $j = \overline{0,n}$ and all $\lambda \quad \mathbb{C}\backslash\{0\}$. Taking into account that element $l_{-1} \quad \mathbb{E}^{n+1}$ depends explicitly both on the vector $x \quad \mathbb{T}^{n+1}$ and on the set of compatible structure matrices $C_k \quad \operatorname{End}\mathbb{E}^{n+1}, k = \overline{0,n}$, it is easy to observe that the system of evolution equations (21) can be equivalently rewritten as

$$\frac{\partial C_k}{\partial t_j} = F_{kj}(C_1, C_2, \dots, C_n) \tag{22}$$

for some $n(n+1)/2 \quad \mathbb{Z}_+$ matrix symmetric functions

$$F_{kj} : (\mathbb{E}^{n+1})^{n+1} \to \operatorname{End}\mathbb{E}^{n+1}, j, k = \overline{0,n},$$

solutions to which solve the initial problem of describing this set of structure matrices, satisfying the oriented associativity equations (3). The obtained results one can formulate as the following proposition.

Proposition 5. *A general solution to the oriented associativity equations* (3)*, provided by the set of evolution equations* (22)*, depends on $n \quad \mathbb{Z}_+$ functional parameters, stemming from a priori nonzero seed component $l_{-1} \quad \mathbb{E}^{n+1}$, entering the evolution flows* (21)*.*

Taking into account the linear representation $l_{-1} = \sum_{s=\overline{0,n}} \bar{l}^s_{-1}(t) x_s$, one can similarly put

$$\varphi_1^{(k)}(l) = \Big\{ \sum_{s=\overline{0,n}} D^s_{kj}(t) x_s : j = \overline{0,n} \Big\} \tag{23}$$

with suitably chosen quadratic matrices $D_k := \big\{D^s_{kj}(t) : j, s = \overline{0,n}\big\}, k = \overline{0,n}$, and obtain a special vector solution

$$l_0 = \Big\{ \sum_{s=\overline{0,n}} \bar{l}^s_{0,j}(t) x_s : j = \overline{0,n} \Big\}, \tag{24}$$

also linear in $x \quad \mathbb{T}^{n+1}$, where the matrix $\bar{l}_0 := \big\{\bar{l}^s_{0,j}(t) : j, s = \overline{0,n}\big\} \quad \operatorname{End}\mathbb{E}^{n+1}$ satisfies, owing to (17), the following matrix equation:

$$\big(\operatorname{tr} C_k\big)\bar{l}_0 + (\operatorname{tr} D_k)\bar{l}_{-1} + C_k^{\intercal}\bar{l}_0 + \bar{l}_0 C_k + D_k^{\intercal}\bar{l}_{-1} + \bar{l}_{-1} D_k = 0 \tag{25}$$

for all $k = \overline{0,n}$. Analytical solutions to the matrix equations (18) and (25) make it possible to construct the seed vector $l = \big(\bar{l}_0 + \lambda^{-1}\bar{l}_{-1}\big)x \quad \mathbb{E}^{n+1}$ and derive the separated evolution equations (22). The realization of this scheme for partial cases $n = 2, 3$, and so on, is postponed for another work in progress.

3. Conclusion

In the work we proposed an effectively enough and algebraically feasible approach to investigating solutions to the oriented associativity equations, related with commutative and isoassociative algebras, interesting for applications in the quantum deformation theory and in some other fields of mathematics. Our construction is based on a version of the Adler–Kostant–Symes scheme, applied to the Lie algebra of the loop diffeomorphism group of a torus, devised recently in [11] and modified for the case of the Gauss–Manin displacement equations, depending on a spectral parameter. Their interpretation as characteristic equations for some system of the Lax–Sato type vector field equations made it possible to apply and suitably develop the results from [11] and derive the determining separated evolution equations for the related structure matrices, generating commutative and associative algebras under regard.

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2+2-Moulton Configuration – rigid and flexible

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Abstract. We consider a new problem on Moulton configuration. For a given two-body system, we add two bodies in a way such that (i) the total four-body system is also in a state of collinear central configuration without changing the positions of original two bodies and (ii) the initial two-body system keeps its motion without any change during the process. We show the existence of this configurations in a general way. We also pose a flexible version of this problem by modifying the condition (ii) and we also find solutions to the second problem.

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1. Introduction

Solutions of Newtonian n-body problem on a line are called a *Moulton configuration*, which will be also abbreviated as M.c., and they become *collinear central configuration*, that is, the ratios of the distances of the bodies from the center of mass are constants [2, 4]. F. R. Moulton [4] proved that for a fixed mass vector $\mathbf{m} = (m_1, \ldots, m_n)$ and a fixed ordering of the bodies along the line, there exists a unique collinear central configuration $\mathbf{q} = (q_1, \ldots, q_n)$ with mass $\mathbf{m} = (m_1, \ldots, m_n)$ (up to translation and scaling), where q_i denotes the position of ith body, $i = 1, \ldots, n$.

In this paper, we consider the following problem [5]. We assume we are given an M.c. $\mathbf{q}_A = (q_{A_1}, q_{A_2})$ of two bodies A_1, A_2 such that $q_{A_1} < q_{A_2}$ with mass $\mathbf{m}_A = (m_{A_1}, m_{A_2})$. We add two bodies B_1, B_2 to A_1, A_2 on the same line so that (i) the configuration of A_1, A_2, B_1 and B_2 is M.c. without changing the original positions of A_1, A_2 and (ii) the motion of A_1, A_2 are kept invariant during the process. More precisely, let q_i $(i = 1, 2, 3, \ )$ denote one of the positions of A_1, A_2, B_1 and B_2, different from each other, and m_i be its mass, respectively.

Definition 1 (2+2-Moulton configuration)**.** We call $\mathbf{q} = (q_1, q_2, q \ , q_4)$ with $\mathbf{m} = (m_1, m_2, m \ , m_4)$ a "2+2-Moulton configuration" for two bodies A_1 and A_2 when it satisfies the following conditions:

(i) A_1, A_2 and B_1 and B_2 are in Moulton configuration and the configuration of A_1, A_2 is equal to the original one $\mathbf{q}_A$ with $\mathbf{m}_A$.

(ii) The center of mass of A_1, A_2, B_1 and B_2 is equal to that of A_1, A_2, and the motion of A_1, A_2 is the same as the original one.

In [5] we have shown an existence of 2+2-M.c. In this paper we give a more general proof of this theorem as follows.

Theorem 2 (2+2-Moulton configuration)**.** *For a given Moulton configuration* $\mathbf{q}_A = (q_{A_1}, q_{A_2})$ *with* $\mathbf{m}_A = (m_{A_1}, m_{A_2})$*,*

(i) *there exist three 2+2-Moulton configurations,*

(ii) *each mass of the added bodies is zero.*

We also consider the situation such that only the condition (i) of Definition 1 is satisfied, namely A_1, A_2, B_1 and B_2 is in Moulton configuration, while the configuration of A_1, A_2 is the same as the original one but their motion is not necessarily equal to the original one. The 2+2-Moulton configuration in Definition 1 can be regarded as a strong or rigid version and the following Definition may give 2+2-Moulton configuration of weak or flexible version. The second main theorem of this paper is on an existence of this general configuration as follows.

Definition 3 (2+2-weak Moulton configuration)**.** We call $\mathbf{q}$ with $\mathbf{m}$ a "2+2-weak Moulton configuration" for $\mathbf{q}_A$ with $\mathbf{m}_A$ when it satisfies only the condition (i) of Definition 1.

Theorem 4. *For a Moulton configuration* $\mathbf{q}_A = (q_{A_1}, q_{A_2})$ *with* $\mathbf{m}_A = (m_{A_1}, m_{A_2})$*, there is a domain of* $\mathbb{R}^2$ *of the configurations of added bodies such that for every configuration of the domain, there exists a 2+2-weak Moulton configuration having strictly positive masses.*

2. Equations for 2+2-Moulton configuration

Collinear central configuration. We consider n bodies lying on a straight line which give a collinear solution of the Newtonian n-body problem. It is known [4] that the ratios of the distances of the bodies from the center of mass are constant. Then we can divide the equation into that of the distances and that of the ratios, and the problem for ratios, namely, the problem of collinear central configuration, is given as follows (cf. [1, 4, 5]). Let $\mathbf{q} = (q_1, \ldots, q_n)$ and $\mathbf{m} = (m_1, \ldots, m_n)$ denote their configuration and mass vector, respectively. We consider a real number c

representing the center of mass and set $\mathbf{c} = (c, \ldots, c) \quad \mathbb{R}^n$. Then $\mathbf{q}$ is a *collinear central configuration*, or a *Moulton configuration,* when it satisfies the equation

$$A \cdot {}^t\mathbf{m} + \lambda^t(\mathbf{q} - \mathbf{c}) = \mathbf{0} \quad \text{for some } \lambda \quad \mathbb{R}, \tag{1}$$

where A is a skew-symmetric matrix defined by $A = (a_{ij}),\ a_{ij} = (q_i - q_j)^{-2}$ for $i < j$, and $a_{ii} = 0$, $a_{ji} = -a_{ij}$. The constant $\sqrt{\lambda}$ denotes an angular velocity of the system (cf. [1, 4]) and determines the equation for the distances (cf. [4, 5]). It is known that any configuration of two bodies is always a collinear central configuration.

Equations for $n = 2$**.** Let $\mathbf{q}_A = (q_{A_1}, q_{A_2})$ be the configuration of two bodies A_1, A_2 and let $\mathbf{m}_A = (m_{A_1}, m_{A_2})$ be its mass vector such that $m_{A_i} > 0$, $i = 1, 2$. We assume that $q_{A_1} < q_{A_2}$. Then equation (1) is written explicitly as

$$\begin{pmatrix} m_{A_1} \\ m_{A_2} \end{pmatrix} = \frac{\lambda_A}{a_{12}} \begin{pmatrix} q_{A_2} - c_A \\ c_A - q_{A_1} \end{pmatrix}, \tag{2}$$

where c_A is the center of mass, satisfying $q_{A_1} < c_A < q_{A_2}$.

Equation for $n =$ **.** Let $\mathbf{q} = (q_1, q_2, q\ , q_4)$ be the configuration of four bodies such that $q_1 < q_2 < q\ < q_4$ and let $\mathbf{m} = (m_1, m_2, m\ , m_4)$ be its mass vector. As to 2+2-Moulton configuration, we consider $m_i \geq 0$, $i = 1, \ldots,\ $. For $n =\ $, the coefficient matrix is invertible and the inverse is given $A^{-1} = B/P$ in (3) below. Then equation (1) is written explicitly as

$$\begin{pmatrix} m_1 \\ m_2 \\ m \\ m_4 \end{pmatrix} = -\frac{\lambda}{P} \begin{pmatrix} 0 & -a\ {}_4 & a_{24} & -a_2 \\ a\ {}_4 & 0 & -a_{14} & a_1 \\ -a_{24} & a_{14} & 0 & -a_{12} \\ a_2 & -a_1 & a_{12} & 0 \end{pmatrix} \begin{pmatrix} q_1 - c \\ q_2 - c \\ q\ - c \\ q_4 - c \end{pmatrix}, \tag{3}$$

where $P = a_{12}a\ {}_4 - a_1\ a_{24} + a_{14}a_2\ $ is the Pfaffian of A.

3. Proof of Theorem 2

Now suppose we are given two bodies A_1, A_2 with a Moulton configuration $\mathbf{q}_A = (q_{A_1}, q_{A_2})$ and the mass $\mathbf{m}_A = (m_{A_1}, m_{A_2})$ such that $q_{A_1} < q_{A_2}$ and $m_{A_1}, m_{A_2} > 0$. We add two bodies B_1, B_2 with masses m_{B_1}, m_{B_2} on the line containing A_1, A_2 so that A_1, A_2, B_1 and B_2 give a 2+2-Moulton configuration for two bodies A_1, A_2.

We set the distance of q_{A_1} and q_{A_2} to one unit and $c_A = 0$ by scaling and translation for simplicity, then equation (2) is written as

$$m_{A_1} = \lambda_A q_{A_2}, \quad m_{A_2} = -\lambda_A q_{A_1} \tag{4}$$

because $a_{12} = (q_{A_2} - q_{A_1})^{-2} = 1$.

Condition (i) of Definition 1. By the condition, the four bodies are in a Moulton configuration and then satisfy equation (3). Using a convention such that $a_{ij} = -a_{ji}$ $(1 \leq i, j \leq \ )$ for the components of the coefficient matrix A, we can write the equation in the form

$$m_i = -\lambda(\alpha_i - c\beta_i)/P, \tag{5}$$

where α_i, β_i are the ith components of $B^t\mathbf{q}$ and $B^t(1,1,1,1)$, respectively, where $B/P = A^{-1}$ (see (3)), and are written as

$$\alpha_i = (-1)^i \sum_{(j,k,l)} a_{jk} q_l, \quad \beta_i = (-1)^i \sum_{(j,k,l)} a_{jk}, \quad 1 \leq i, j, k, l \leq \ ,$$

and $\sum_{(j,k,l)}$ is a cyclic summation.

Now we suppose A_1 is the ith body and A_2 is the jth body among $q_i, i = 1, \dots, \ $. Then $m_{A_1} = m_i$, $m_{A_2} = m_j$ $(1 \leq i < j \leq \ )$, and thus by (4) condition (i) is equivalent to the equation

$$\begin{pmatrix} m_i \\ m_j \end{pmatrix} = -\frac{\lambda}{P}\left(\begin{pmatrix} \alpha_i \\ \alpha_j \end{pmatrix} - c\begin{pmatrix} \beta_i \\ \beta_j \end{pmatrix}\right) = \begin{pmatrix} m_{A_1} \\ m_{A_2} \end{pmatrix} = \lambda_A \begin{pmatrix} q_j \\ -q_i \end{pmatrix}. \tag{6}$$

From this equation we obtain

$$c = \frac{\alpha_i q_i + \alpha_j q_j}{\beta_i q_i + \beta_j q_j}, \qquad \lambda = \lambda_A P \frac{\beta_i q_i + \beta_j q_j}{\alpha_j \beta_i - \alpha_i \beta_j}.$$

Then equation (5) yields the masses of the added bodies are given as

$$m_k = \frac{\lambda_A}{\alpha_j \beta_i - \alpha_i \beta_j}\left((\alpha_i q_i + \alpha_j q_j)\beta_k - (\beta_i q_i + \beta_j q_j)\alpha_k\right), \quad k \neq i, j. \tag{7}$$

Condition (ii) of Definition 1. This condition is written explicitly as $c = 0$ and $\lambda = \lambda_A$, which gives the equations

$$\alpha_i q_i + \alpha_j q_j = 0, \qquad P(\beta_i q_i + \beta_j q_j) = \alpha_j \beta_i - \alpha_i \beta_j. \tag{8}$$

Thus, B_i $(i = 1, 2)$ give a 2+2-Moulton configuration if and only if (q_{B_1}, q_{B_2}) satisfies equations (6) and (8).

We remark here that by a simple calculation equation (8) gives $\alpha_k = \alpha_l = 0$, which plays an important role afterwards.

Proof of Theorem 2. Now we discuss the existence of q_{B_1}, q_{B_2} satisfying equations (6) and (8) in the following six cases:

Case 1: $q_{B_1} < q_{B_2} < q_{A_1} < q_{A_2}$; Case 2: $q_{B_1} < q_{A_1} < q_{B_2} < q_{A_2}$;
Case 3: $q_{B_1} < q_{A_1} < q_{A_2} < q_{B_2}$; Case 4: $q_{A_1} < q_{B_1} < q_{B_2} < q_{A_2}$;
Case 5: $q_{A_1} < q_{B_1} < q_{A_2} < q_{B_2}$; Case 6: $q_{A_1} < q_{A_2} < q_{B_1} < q_{B_2}$.

First, a direct calculation gives the following.

Lemma 5. *Equation* (8) *yields* $\alpha_i = -Pq_j, \quad \alpha_j = Pq_i$.

Case 1: $q_{B_1} < q_{B_2} < q_{A_1} < q_{A_2}$. In this case $(q_1, q_2, q\ , q_4) = (q_{B_1}, q_{B_2}, q_{A_1}, q_{A_2})$. Applying Lemma 5 gives $-Pq_4 = \alpha\ = (-1)\ (a_{41}q_2 + a_{12}q_4 + a_{24}q_1)$, $Pq\ = \alpha_4 = (-1)^4(a_{12}q\ + a_2\ q_1 + a\ {}_1q_2)$, and then we have $a_{24}q_1 - a_{14}q_2 = (P - a_{12})q_4$ and $a_2\ q_1 - a_1\ q_2 = (P - a_{12})q\ $. From these relations we easily obtain $q_{B_1} = q_1 = a_{14}q\ - a_1\ q_4$, $q_{B_2} = q_2 = a_{24}q\ - a_2\ q_4$, which yield $q_{B_1} = q_{B_2}$. Actually q_{B_1}, q_{B_2} are the solutions of the following equation:

$$f(x) = x - \frac{q_{A_1}}{(x - q_{A_2})^2} + \frac{q_{A_2}}{(x - q_{A_1})^2} = 0, \quad x < q_{A_1} < q_{A_2}.$$

Notice $q_{A_1} < c_A = 0 < q_{A_2}$ and $f(x)$ is strictly increasing for $x < q_{A_1}$. Since $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to q_{A_1}, x<q_{A_1}} f(x) = +\infty$, we obtain the unique solution for $f(x) = 0, x < q_{A_1}$, and this induces $q_{B_1} = q_{B_2}$ which contradicts $q_{B_1} \neq q_{B_2}$. Therefore we have no 2+2-Moulton configuration in case 1. We can show similar result in case 4 and 6, and thus no 2+2-Moulton configurations for these cases.

Case 2: $q_{B_1} < q_{A_1} < q_{B_2} < q_{A_2}$. In this case $(q_1, q_2, q\ , q_4) = (q_{B_1}, q_{A_1}, q_{B_2}, q_{A_2})$ and similarly as in case 1, we obtain $q_{B_1} = q_1 = a_{14}q_2 - a_{12}q_4$, $q_{B_2} = q\ = a\ {}_4q_2 + a_2\ q_4$. Note here that $q_1 = a_{14}q_2 - a_{12}q_4$, $q\ = a\ {}_4q_2 + a_2\ q_4$ are equivalent to $\alpha\ = 0$, $\alpha_1 = 0$, respectively, since $a_{24} = 1$.

Thus q_{B_1}, q_{B_2} are respectively solutions of the following equations:

$$f_{21}(x) = x - \frac{q_{A_1}}{(x - q_{A_2})^2} + \frac{q_{A_2}}{(x - q_{A_1})^2} = 0, \quad x < q_{A_1},$$

$$f_2\ (x) = x - \frac{q_{A_1}}{(x - q_{A_2})^2} - \frac{q_{A_2}}{(q_{A_1} - x)^2} = 0, \quad q_{A_1} < x < q_{A_2}.$$

These functions are strictly increasing because $df_{21}/dx > 0$ for $x < q_{A_1}$, $df_2\ /dx > 0$ when $q_{A_1} < x < q_{A_2}$, and $\lim_{x\to-\infty} f_{21} = -\infty$, $\lim_{x\to q_{A_1}} f_{21} = +\infty$, and $\lim_{x\to q_{A_1}} f_2\ = -\infty$ $\lim_{x\to q_{A_2}} f_2\ = +\infty$. Then there is a unique solution (q_{B_1}, q_{B_2}) for the equation above and which gives a 2+2-Moulton configuration. Moreover, we substitute $\alpha_i q_i + \alpha_j q_j = 0$, $\alpha_k = 0$ $(k = 1, 3)$ into equation (7), and then we obtain $m_{B_i} = 0$ $(i = 1, 2)$. Thus we obtain Theorem 2 for case 2.

Case 3: $q_{B_1} < q_{A_1} < q_{A_2} < q_{B_2}$. In this case $(q_1, q_2, q\ , q_4) = (q_{B_1}, q_{A_1}, q_{A_2}, q_{B_2})$. In the same way as in the previous cases we obtain $q_{B_1} = q_1 = a_1\ q_2 - a_{12}q\ $, $q_{B_2} = q_4 = a_{24}q\ - a\ {}_4q_2$. Then q_{B_1}, q_{B_2} are the solutions of the following equations

$$f\ {}_1(x) = x - \frac{q_{A_1}}{(x - q_{A_2})^2} + \frac{q_{A_2}}{(x - q_{A_1})^2} = 0, \quad x < q_{A_1},$$

$$f\ {}_4(x) = x - \frac{q_{A_2}}{(q_{A_1} - x)^2} + \frac{q_{A_1}}{(q_{A_2} - x)^2} = 0, \quad q_{A_2} < x.$$

The functions $f\ {}_1(x)$, $f\ {}_4(x)$ are strictly increasing, which yields a unique solution (q_{B_1}, q_{B_2}) for these equations, and then we have a unique 2+2-M.c..

Case 5: $q_{A_1} < q_{B_1} < q_{A_2} < q_{B_2}$. In this case we also have a unique 2+2-M.c. by means of the same argument as in cases 1 and 3.

For cases 3 and 5 we can also show $\alpha_k = 0$, and then by using (7) we obtain Theorem 2.

4. Proof of Theorem 4

In the previous sections, the masses of the added bodies are given as functions of q_{B_1}, q_{B_2}, namely, $m_{B_i} = m_{B_i}(q_{B_1}, q_{B_2})$ $(i = 1, 2)$. Then (q_{B_1}, q_{B_2}) together with $(m_{B_1}(q_{B_1}, q_{B_2}), m_{B_2}(q_{B_1}, q_{B_2}))$ is a 2+2-weak Moulton configuration.

In this section, we show there are open domains such that for every (q_{B_1}, q_{B_2}) of the domain, the masses $m_{B_i}(q_{B_1}, q_{B_2})$, $(i = 1, 2)$ are strictly positive. For this purpose, taking account of (7), it is enough to consider the following functions:

$$\begin{aligned}
\tilde{m}_{B_1}(q_{B_1}, q_{B_2}) &= \tilde{m}_k(q_k, q_l) = c(q_k, q_l)\beta_k(q_l) - \alpha_k(q_l), \\
\tilde{m}_{B_2}(q_{B_1}, q_{B_2}) &= \tilde{m}_l(q_k, q_l) = c(q_k, q_l)\beta_l(q_k) - \alpha_l(q_k) \quad (1 \le k < l \le \quad).
\end{aligned}$$

We notice that the function α_k depends only on q_l and then $\frac{\partial \alpha_k}{\partial q_k}(q_l) = 0$. Also remark $m_{B_i} > 0$ if and only if $\tilde{m}_{B_i} > 0$ since $\lambda/P > 0$.

Now we consider the Jacobian of $J = \det \begin{pmatrix} \partial \tilde{m}_k/\partial q_k & \partial \tilde{m}_k/\partial q_l \\ \partial \tilde{m}_l/\partial q_k & \partial \tilde{m}_l/\partial q_l \end{pmatrix}$. Then it suffices to show $J \neq 0$ at the point (q_{B_1}, q_{B_2}) such that $c = c_A = 0$ because this yields the existence the desired domain in the neighborhood of 2+2-Moulton configuration. Using $c = 0$, $\frac{\partial \alpha_k}{\partial q_k}(q_l) = 0$ and $\frac{\partial \alpha_l}{\partial q_l}(q_k) = 0$, we obtain

$$J = \frac{\partial \alpha_k}{\partial q_l}(q_l)\left(\frac{\partial c}{\partial q_k}(q_k, q_l) \cdot \beta_l - \frac{\partial \alpha_l}{\partial q_k}(q_k)\right) + \frac{\partial c}{\partial q_l}(q_k, q_l) \cdot \frac{\partial \alpha_l}{\partial q_k}(q_k) \cdot \beta_k.$$

We note here that $\alpha_i q_i + \alpha_j q_j = -\alpha_k q_k - \alpha_l q_l$ because $\mathbf{q}\,{}^t(\alpha_1, \alpha_2, \alpha\ \ , \alpha_4) = \mathbf{q} B\,{}^t\mathbf{q} = 0$ since B is skew-symmetric matrix $B = A^{-1}P$, and when $c = 0$, we have $\alpha_i q_i + \alpha_j q_j = 0$ and $\alpha_k = \alpha_l = 0$ from equation (8). Therefore we obtain

$$\frac{\partial c}{\partial q_k} = \frac{1}{\beta_i q_i + \beta_j q_j}\left(\frac{\partial}{\partial q_k}(-\alpha_k q_k - \alpha_l q_l)\right) = \frac{-\alpha_l^{(k)} q_l}{\beta_i q_i + \beta_j q_j},$$

where $\alpha_l^{(k)} = \partial \alpha_l / \partial q_k$ and so on. Similarly, the numerator of $\partial c/\partial q_l$ is equal to $-\alpha_k^{(l)} q_k$. Then substituting these into the right side of the Jacobian J gives $J = -\frac{\alpha_k^{(l)}\alpha_l^{(k)}}{\beta_i q_i + \beta_j q_j}\sum_{t=1}^4 \beta_t q_t \neq 0$ because $\alpha_k^{(l)} \neq 0$, $\alpha_l^{(k)} \neq 0$ and $\sum_{t=1}^4 \beta_t q_t > 0$.

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Melnikov functions in the rigid body dynamics

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Abstract. We review our recent results about perturbations of two cases in the rigid body dynamics: the Hess–Appelrot case and the Lagrange case.

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1. The Hess–Appelrot and Lagrange cases

The **Euler–Poisson system** (EP system) system

$$\dot{\boldsymbol{M}} = \boldsymbol{M} \times \boldsymbol{\Omega} - \boldsymbol{\Gamma} \times \boldsymbol{K}, \qquad \dot{\boldsymbol{\Gamma}} = \boldsymbol{\Gamma} \times \boldsymbol{\Omega} \tag{1}$$

describes the motion of a rigid body in a coordinate system associated with the body (see [3]). Above $\boldsymbol{M} = I\boldsymbol{\Omega} = (I_1\Omega_1, I_2\Omega_2, I\ \Omega\ )$ is the angular momentum, $\boldsymbol{\Omega}$ is the angular velocity, $\boldsymbol{\Gamma}$ is the unit vector in the direction of the gravity force and $\boldsymbol{K}$ is a constant vector associated with the center of mass of the body.

System (1) is Hamiltonian with respect to certain Poisson structure with the **Casimir functions** $|\boldsymbol{\Gamma}|^2$ (which we assume equal to 1) and $(\boldsymbol{\Gamma}, \boldsymbol{M})$ (the minus vertical component of the angular momentum, $-m\ $) with the total energy $H = \frac{1}{2}(\boldsymbol{M}, \boldsymbol{\Omega}) - (\boldsymbol{K}, \boldsymbol{\Gamma})$ (equal to E) as the Hamilton function.

The **Hess–Appelrot case** (HA case) is defined by the following conditions:

$$K_2 = 0, \quad K_1\sqrt{I_1(I_2 - I\ )} = K\ \sqrt{I\ (I_1 - I_2)}, \tag{2}$$

The peculiarity of this case relies upon the existence of the so-called **Hess surface**

$$\mathcal{S} = \{(\boldsymbol{K}, \boldsymbol{M}) = 0\} \tag{3}$$

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invariant for the EP system. Moreover, one finds that, after restriction to $\mathcal{S}$, the kinetic energy $\frac{1}{2}(\boldsymbol{M}, \boldsymbol{\Omega})$ equals $|\boldsymbol{M}|^2/2I_2$. Next, by expanding the vector $\boldsymbol{\Gamma}$ in the orthogonal frame $\{\boldsymbol{M}, \boldsymbol{K}, \boldsymbol{M} \times \boldsymbol{K}\}$ (i.e., on $\mathcal{S}$) and using the above properties, one arrives at the following algebraic-differential system

$$\dot{x} = 2y, \quad y^2 = R(x), \tag{4}$$

where

$$x = |\boldsymbol{M}|^2, \quad y = (\boldsymbol{M}, \boldsymbol{\Gamma} \times \boldsymbol{K})$$

and $R(x)$ is a cubic polynomial. The latter equations are integrated in terms of elliptic functions: $x = \mathcal{P}(t)$ is the Weierstrass $\mathcal{P}$-function. It is periodic with the period $T_1 = \frac{1}{2} \oint_\gamma \mathrm{d}x/y$, where γ is an oval of the elliptic curve defined in the second equation of Eq. (4).

System (4) is supplemented with the following equation for the angle-type variable

$$u = \tan(\varphi/2) \quad \mathbb{RP}^1,$$

$\varphi = \arctan\left(\frac{|\boldsymbol{K}|}{K}\frac{M_2}{M_1}\right)$:

$$\dot{u} = A(x) + B(x)u^2, \tag{5}$$

where $A = b/x + a\sqrt{x}$, $B = b/x - a\sqrt{x}$ for some constants a, b. One can also see that the Hess surface is a torus.

Inserting the solution $x = \mathcal{P}(t)$ into Eq. (5) one obtains a periodic Riccati equation with the **monodromy map** (evolution after the period) $\mathcal{M}(u) = \frac{\alpha u+\beta}{\gamma u+\delta}$. Depending on the situation, the dynamics can be **hyperbolic** (with two T_1-periodic solutions), **parabolic**, **elliptic quasi-periodic** or **elliptic periodic** (with a second period $T_2 = \frac{q}{p}T_1$, $p, q \quad \mathbb{N}$, $\gcd(p, q) = 1$).

The **Lagrange case** lies at the 'boundary' of the HA case; namely, assuming $K_1 \to \infty$ and $I_2 - I_1 \to 0$ in Eq. (2), we arrive at the symmetric top conditions (see [7]):

$$K_1 = K_2 = 0, \quad I_2 = I_1. \tag{6}$$

Now the **Hess function** $(\boldsymbol{K}, \boldsymbol{M})$ is invariant (equal to $K \ M$). So, we have a completely integrable system (in any symplectic leaf $\{|\boldsymbol{\Gamma}|^2 = 1, \ (\boldsymbol{M}, \boldsymbol{\Gamma}) = -m \ \}$) with a family of 2-dimensional tori

$$\mathcal{S}_M \ = \{(\boldsymbol{K}, \boldsymbol{M}) = K \ M \ \}. \tag{7}$$

One has the following analogue of Eqs. (4)–(5):

$$\dot{x} = 2y, \quad y^2 = R(x), \quad \dot{\varphi} = D(x), \tag{8}$$

where

$$x = |M|^2, \quad \varphi = \arg M,$$

$M = M_1 + \mathrm{i}M_2 \quad \mathbb{C}$, $R(x)$ is a cubic polynomial and $D(x)$ is a rational functions (see [7]). As before, the solution $x = \mathcal{P}(t)$ is periodic with period T_1 (defined by analogous contour integral but along an oval γ_M). The third equation of Eq. (8) has the solution

$$\varphi(t) = \varphi_* + \int_0^t D(\mathcal{P}(s)\mathrm{d}s.$$

So, we have the second period $T_2 = \int_0^{T_1} D(\mathcal{P}(t))\mathrm{d}t = \frac{1}{2}\oint \frac{D(x)}{y}\mathrm{d}x$.

Finally, one introduces **action–angle variables** (I_j, φ_j), where

$$\varphi_1 = \frac{\pi}{T_1}\int \frac{\mathrm{d}x}{y}, \quad \varphi_2 = \frac{\pi}{T_2}\left(\varphi - \frac{1}{2}\int D(x)\frac{\mathrm{d}x}{y}\right) + \frac{T_1}{T_2}\varphi_1. \tag{9}$$

Above, the integration path lies in the oval γ_M , starts at $(x_1, 0)$ and ends at (x, y).

2. Limit cycles for perturbations of the Hess–Appelrot case

We consider perturbations of the HA case, while remaining in the scope of the EP equations. Thus the HA conditions (2) are used to define two small parameters

$$\varepsilon_1 = \frac{K_2}{|\boldsymbol{K}|}, \quad \varepsilon_2 = \frac{K_1^2}{|\boldsymbol{K}|^2}\left(\frac{1}{I} - \frac{1}{I_2}\right) - \frac{K^2}{|\boldsymbol{K}|^2}\left(\frac{1}{I_2} - \frac{1}{I_1}\right). \tag{10}$$

We have the following equation for the evolution of the **Hess function** $z = (\boldsymbol{K}, \boldsymbol{M})$:

$$\dot{z} = fM_2z - \varepsilon_1(G_1M_1^2 + G_2M_2^2) + \varepsilon_2FM_1M_2 + \cdots, \tag{11}$$

where f, F, G_1, G_2 are constants.

By the normal hyperbolicity theory, the Hess surface $\mathcal{S} = \{z = 0\}$ survives the perturbation when the transversal to it contraction or expansion is stronger than the corresponding dynamics on $\mathcal{S}$. In the case the unperturbed Hess surface supports periodic elliptic dynamics, with the $(p : q)$ resonance between principal periods, the **normal hyperbolicity property** is measured by the following **Lyapunov exponent**:

$$\Lambda = f\int_0^T M_2(t)\mathrm{d}t, \tag{12}$$

where $T = qT_2$ is the period.

In the normal hyperbolicity case the equation for the perturbed invariant surface $\mathcal{S}_\varepsilon$ takes the form $z = \varepsilon_1 z_1 + \varepsilon_2 z_2 + \cdots$, where the functions $z_{1,2} = z_{1,2}(x, y, u)$ are expressed via some integral formulas (which we do not present here).

We focus our attention on the so-called **critical circle subcase** of the HA case, when the oval γ of the elliptic curve $y^2 = R(x)$ is degenerated to a point $(x, y) = (x_+, 0)$. The near ovals are of the form

$$y^2 + k^2\hat{x}^2 = r^2, \tag{13}$$

$\hat{x} = x - x_+ + \cdots$, where the 'radius' $r \geq 0$ is small and k is a constant (see [5]). We introduce the angle type variables on the Hess surface by the formulas

$$\phi = 2\arctan(u\sqrt{B(x_+)/A(x_+)}), \quad \psi = \arctan(r\hat{x}/y); \tag{14}$$

then we obtain the system

$$\begin{aligned}\dot{\psi} &= \nu + ar\sin\psi + \cdots\,,\\ \dot{\varphi} &= \mu + r(b + c\cos\phi)\sin\psi + \cdots\,,\end{aligned} \tag{15}$$

where μ and ν are limit frequencies (provided the dynamics is elliptic) and a, b, c are constants. It is solved by expansion in powers of r.

In [5] the expansion of the Lyapunov exponent (12) was computed. It turns out that $\Lambda = O(r^2)$ when the resonance $(p : q) \neq (1 : q)$ and in the case of $(1 : q)$ resonance we have

$$\Lambda = \mathrm{const}\cdot r\cos\psi_* + \cdots\,, \tag{16}$$

for the initial condition $\psi(0) = \psi_*$ of the periodic trajectory. It means that we have the normal hyperbolicity outside two trajectories.

The vector field restricted to the perturbed invariant surface $\mathcal{S}_\varepsilon$ is not integrable and has some **limit cycles**. The condition for these limit cycles is the following equation for their initial conditions $\psi_* = \psi_*^{(j)}$:

$$J(\psi_*) = \varepsilon_1 J_1(\psi_*) + (\varepsilon_2/r)J_2(\psi_*) = 0. \tag{17}$$

Here

$$J_{1,2} = -\int_0^{2q\pi} \ln|\cos\psi|\cdot\Xi''(\psi)\mathrm{d}\psi,$$

are **Melnikov type integrals** with

$$\begin{aligned}\Xi_1 &= A\left(\frac{1-\rho\cos\phi}{\rho-\cos\phi}\right)^2 + B\cos\psi_*\cdot\Xi_2,\\ \Xi_2 &= \frac{1}{\cos\psi_*}(\rho-\cos\phi)^{\kappa-1}(C - D\cos\phi), \quad \phi = (\phi-\phi_*)/q,\end{aligned}$$

where $A, B, C, D, \varrho, \kappa$ are some constants.

Theorem 1. *The number of zeroes of the function in* (17) *is uniformly bounded from the above by a constant not depending on the parameters of the problem.*

Moreover, it obeys the following partial bound:

$$\leq 56 + 6N(\kappa), \tag{18}$$

where $N(\kappa) = \lceil -2(\kappa+1)\rceil \quad (\kappa < -1), \quad = 0 \quad (-1 \leq \kappa \leq 0), \quad \lceil \kappa/q \rceil \quad (\kappa > 0).$

The second statement of this theorem was proved in [6] using extension of the functions $J_{1,2}$ to the complex domain and applying the argument principle. Note that this estimate depends on the parameter κ and is unbounded. The existence of a uniform bound was proved in [7].

3. Perturbations of the Lagrange case

Recall that the KAM theory predicts that most of the invariant tori of a completely integrable system survive a perturbation (in the class of Hamiltonian systems). In the Lagrange case the survived tori are among those with irrational ratio T_2/T_1 of the principal periods.

In [7] the problem of perturbations of resonant tori, i.e., those with rational $T_2/T_1 = q/p$, was considered. The perturbation parameters are following:

$$\varepsilon_1 = K_1, \quad \varepsilon_2 = K_2, \quad \varepsilon \ = 1/I_2 - 1/I_1. \tag{19}$$

Denote also

$$\epsilon = \varepsilon_1 + \mathrm{i}\varepsilon_2 = |\epsilon| \mathrm{e}^{\mathrm{i}\theta} \quad \mathbb{C}.$$

It is expected that some isolated periodic trajectories of period $T = qT_1$ will appear. We have found that these periodic trajectories correspond to zeroes $\varphi_* = \varphi_*^{(j)}$ of the following **Melnikov type integral**

$$\mathcal{I}(\varphi_*) = C_1 \, |\epsilon| \int_0^T \sqrt{\mathcal{P}(t)} \sin(\varphi(t) - \theta) + C_2 \varepsilon \int_0^T \mathcal{P}(t) \sin 2\varphi(t), \tag{20}$$

where $\varphi(t) = \varphi_* + \int_0^t D(\mathcal{P}(s))\,\mathrm{d}s$ and $x = \mathcal{P}(t)$ is the solution to the equation $\dot{x} = 2\sqrt{R(x)}$ (via the Weierstrass elliptic function) and $C_{1,2}$ are constants.

Moreover, if $\mathcal{I}(\varphi_*^{(j)}) > 0$ then the corresponding solution is **hyperbolic** (of saddle type) and, if $\mathcal{I}(\varphi_*^{(j)}) < 0$ then it is **elliptic**.

Finally, we considered the situation near the **critical circle subcase**, i.e., when the oval γ_M is a small ellipse around a point $(x_+, 0)$.

Theorem 2. *Under some generic assumptions, perturbations of the Lagrange case near the critical circle subcase generate isolated periodic solutions out of $p:q$ resonant tori only in the cases of $1:1$ and $1:2$ resonances modulo $O(r^2)$ and modulo $O(|\varepsilon|^2)$, where r is the smaller 'radius' of the torus. Moreover, there are only two such periodic trajectories, one hyperbolic and one elliptic.*

The analysis suggests that, using first order (linear in ε) Melnikov integral (20), only $p:1$ and $p:2$ resonant periodic trajectories can be found. It is expected that other types of periodic orbits could be revealed using higher order Melnikov functions.

4. Chaotic dynamics for perturbation of separatrix connection subcase of the HA case

Here we consider another situation in the HA case, where calculations can be carried out explicitly. This is the so-called **separatrix connection subcase**, which corresponds to the situation when the oval γ becomes a loop with vertex at $(x, y) = (0, 0)$ (see [2, 4]). For the 4-dimensional system (in the symplectic leaf) the latter vertex corresponds to a singular point O with a pair of real eigenvalues $\pm\sigma$, $\sigma =$

$\sqrt{|\boldsymbol{K}|/I_2}$, of multiplicity two each and with trivial Jordan form. The point O has two 2-dimensional separatrices: stable $\mathcal{W}^s$ and unstable $\mathcal{W}^u$. Globally, they coincide and form a **separatrix connection** $\mathcal{S}$, which is the Hess surface.

The solutions $x(t)$, $u(t)$ (for x, u defined in Section 1), which lie in $\mathcal{S}$, are given by

$$x = \frac{c^2}{\cosh^2 \mu t}, \quad \frac{u-1}{u+1} = q \left(\frac{\mathrm{e}^{-\mu t} + \mathrm{i}}{\mathrm{e}^{-\mu t} - \mathrm{i}} \right)^{\mathrm{i}\nu}, \tag{21}$$

where $q = \frac{u(-\infty)-1}{u(-\infty)+1} \quad \mathbb{RP}^1$ measures the initial direction in $\mathcal{W}^u$ of the corresponding phase curve in $\mathcal{S}$ and c, μ, ν are constants.

Consider a perturbation of this situation. One parameter is

$$\varepsilon_0 = (\boldsymbol{M}, \boldsymbol{K}) \tag{22}$$

and other are like in Eq. (10).

In order to study the dynamics, we consider a Poincaré type map defined as follows. Take two sections to the phase curves: Δ transversal to $\mathcal{W}^u$ and Σ transversal to $\mathcal{W}^s$. They are of the form $\mathbb{S}^1 \times \mathbb{D}^2$. Let $\mathcal{Q} : \Sigma \mapsto \Delta$ and $\mathcal{R} : \Delta \mapsto \Sigma$ be the natural maps defined by intersection of the phase curves with these sections. The **Poincaré map** is $\mathcal{T} = \mathcal{Q} \circ \mathcal{R}$.

The map $\mathcal{R}$ measures the splitting of the separatrix connection $\mathcal{S}$. There remain only finitely many 1-dimensional **homoclinic trajectories** δ_{q_j}, such that the initial directions q_j are zeroes of the following **Melnikov function**:

$$\mathcal{I}(q) = \varepsilon_0 \mathcal{I}_0(q) + \varepsilon_1 \mathcal{I}_1(q) + \varepsilon_2 \mathcal{I}_2(q), \tag{23}$$

where

$$\begin{aligned}
\mathcal{I}_0 &= |\boldsymbol{K}| \left\{ \exp\left(f \int M_2(t;q)\mathrm{d}t \right) - 1 \right\}, \\
\mathcal{I}_1 &= \int \left(F_1 M_1^2(t;q) + F_2 M_2^2(t;q) \right) \mathrm{d}t, \\
\mathcal{I}_2 &= G \int M_1(t;q) M_2(t;q) \mathrm{d}t,
\end{aligned}$$

where the constants f, F_1, F_2, G are the same as in Eq. (11), the integrations are performed along the infinite interval $(-\infty, \infty)$ and

$$\begin{aligned}
M_1(t;q) &= \frac{K}{K} \frac{c}{\cosh \mu t} \frac{-2q\xi(t)^{\mathrm{i}\nu}}{1 + q^2 \xi(t)^{2\mathrm{i}\nu}}, \\
M_2(t;q) &= \frac{c}{\cosh \mu t} \frac{1 - q^2 \xi(t)^{2\mathrm{i}\nu}}{1 + q^2 \xi(t)^{2\mathrm{i}\nu}}, \\
\xi(t) &= \frac{e^{-\mu t} + \mathrm{i}}{e^{-\mu t} - \mathrm{i}}.
\end{aligned}$$

The map $\mathcal{R}$ (defined via trajectories near O) is responsible for hyperbolic properties of the Poincaré map. The eigenvalues of the singular point after perturbation take the form

$$\lambda = \pm\sigma \pm \sqrt{D(\varepsilon)}, \quad D(\varepsilon) = \tilde{\varepsilon}_2^2 + \ \tilde{\varepsilon}_1^2 - \ \tilde{\varepsilon}_0^2, \tag{24}$$

where $\tilde{\varepsilon}_j = \mathrm{const} \cdot \varepsilon_j$.

R. Devaney [1] proved that, if the eigenvalues have nonzero imaginary part and there exists a homoclinic connection, then the Poincaré return map exhibits a chaotic dynamics (there exists a Smale-type horseshoe). Of course, the first requirement is satisfied when $\varepsilon_1 = \varepsilon_2 = 0 \neq \varepsilon_0$. One can see that in this case the Melnikov integral (23) has at least one zero.

Based on this, in [4] we proved the following

Theorem 3. *For ε from an open and nonempty cone in $\mathbb{R}$ with vertex at $\varepsilon = 0$ there exists a compact subset $\Lambda \subset \Delta$, which is invariant and hyperbolic for $\mathcal{T}$, and $T\,|_\Lambda$ is topologically conjugate with the Bernoulli shift on $N \geq 2$ symbols.*

This implies, in particular, that the EP system near the separatrix connection subcase of the HA case, i.e., for ε from the above subset, does not admit any additional first integral depending analytically on the coordinates.

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Part II

Quantization

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$\mathbf{E}(2)$-covariant integral quantization of the motion on the circle and its classical limit

Rodrigo Fresneda, Jean Pierre Gazeau and Diego Noguera

Abstract. The present contribution is based on the work [1], where the motion of a particle on a circle is studied using the general method of covariant integral quantization of semi-direct product groups for the special case of E(2). We have added to the original work the analysis of the classical limit.

Mathematics Subject Classification (2000). 81R30; 81S30;81S10.

Keywords. Euclidean group, Coherent states, Covariant quantization, Quantum angle, Classical limit.

1. Introduction

The aim of this contribution is to present an elementary illustration of covariant integral quantization in the case of semi-direct product groups. In Section 2 we make a brief description of the method and we then specify to the group E(2). We first build coherent states for UIR (unitary irreducible representation) of semi-direct product groups which are square integrable modulo a subgroup. Then, we implement covariant CS (coherent states) integral quantization for the motion of a particle on the circle. In this case, the phase space is the cylinder viewed as a left coset of the Euclidean group E(2). Coherent states issued from fiducial vectors are labeled by points in the cylinder and also depend on extra parameters. In Section 3 we describe the covariance of the quantization map. In Section 4 we consider some examples, such as the quantizations of the angular momentum and the calculation of lower symbols and the classical limit (see Theorem 6). In Section 5, we carry out the quantization of the 2π-periodic discontinuous angle function, and calculate its commutator with the angular momentum and the resulting Heisenberg inequality. Notwithstanding the classical limit, all proofs are omitted, as well as the references, which can all be found in [1].

2. Covariant integral quantizations

Let $X \sim G/H$ be a homogeneous space viewed as the left coset manifold for the action of a Lie group G, where subgroup H is the stabilizer of some point of X. If X is a symplectic manifold (e.g., a co-adjoint orbit of G), it may be viewed as the phase space for the dynamics of some system. Let ν be a quasi-invariant measure on X, that is, $\mathrm{d}\nu(g^{-1}x) = \lambda(g,x)\mathrm{d}\nu(x)$ $(\forall g \in G)$ with $\lambda(g_1g_2,x) = \lambda(g_1,x)\,\lambda(g_2,g_1^{-1}x)$ (a cocycle). For a global Borel section $\sigma : X \to G$ of the group, let ν_σ be the unique quasi-invariant measure defined by $\mathrm{d}\nu_\sigma(x) = \lambda(\sigma(x),x)\,\mathrm{d}\nu(x)$. We say that a UIR is square-integrable $\mod (H,\sigma)$ if there exists a density operator ρ, i.e., $\rho \geq 0$, $\mathrm{Tr}(\rho) = 1$, such that

$$c_\rho := \int_X \mathrm{tr}\left(\rho\,\rho_\sigma(x)\right)\,\mathrm{d}\nu_\sigma(x) < \infty$$

with $\rho_\sigma(x) := U(\sigma(x))\rho U(\sigma(x))^\dagger$. Then *square-integrability* $\mathrm{mod}(H,\sigma)$ gives

$$I = \frac{1}{c_\rho}\int_X \rho_\sigma(x)\,\mathrm{d}\nu_\sigma(x).$$

For $\rho = |\eta\rangle\langle\eta|$, $|\eta_x\rangle := |U(\sigma_g(x))\eta\rangle$ is a coherent state and η is a “fiducial” vector.

The resolution of the identity allows the covariant integral quantization of functions (with possible extension to distributions) on X given by the expression

$$f \mapsto A^\sigma_f = \frac{1}{c_\rho}\int_X f(x)\,\rho_\sigma(x)\,\mathrm{d}\nu_\sigma(x).$$

Consider sections $\sigma_g : X \to G$, $g \in G$, which are covariant translates of σ under g: $\sigma_g(x) = g\sigma(g^{-1}x) = \sigma(x)h(g,g^{-1}x)$, where the cocycle $h(g,x)$ belongs to H. Given $\mathrm{d}\nu_{\sigma_g}(x) := \lambda(\sigma_g(x),x)\,\mathrm{d}\nu(x)$, define $\rho_{\sigma_g}(x) = U(\sigma_g(x))\rho U(\sigma_g(x))^\dagger$. Then, for U square-integrable $\mathrm{mod}(H,\sigma)$, the general covariance property of the integral quantization $f \mapsto A^\sigma_f$ reads $U(g)A^\sigma_f U(g)^\dagger = A^{\sigma_g}_{\mathcal{U}_l(g)f}$, where

$$A^{\sigma_g}_f := \frac{1}{c_\rho}\int_X \rho_{\sigma_g}(x)f(x)\mathrm{d}\nu_{\sigma_g}(x) \quad \text{with } \mathcal{U}_l(g)f(x) = f\left(g^{-1}x\right).$$

If $\rho = |\eta\rangle\langle\eta|$, we are working with CS quantization.

2.1. Illustration with E(2)

The group $\mathrm{E}(2) = \mathbb{R}^2 \rtimes \mathrm{SO}(2) = \{(\mathbf{r},\theta)\,,\, \mathbf{r} \in \mathbb{R}^2\,,\, \theta \in [0,2\pi)\}$ is equipped with the composition rule $(\mathbf{r},\theta)(\mathbf{r}',\theta') = (\mathbf{r} + \mathcal{R}(\theta)\mathbf{r}', \theta + \theta')$ and inverse $(\mathbf{r},\theta)^{-1} = (-\mathcal{R}(-\theta)\mathbf{r}, -\theta)$; $\mathcal{R}(\theta) = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$ rotates vectors in the plane by angle θ.[1] We denote by $L^2(\mathbb{S}^1, \mathrm{d}\alpha)$ the Hilbert space of 2π-periodic complex-valued functions $\psi(\alpha)$ which are square-integrable on a period interval $[\alpha_0, \alpha_0 + 2\pi]$, $\alpha_0 \in \mathbb{R}$,

$$\int_{\alpha_0}^{\alpha_0+2\pi} \mathrm{d}\alpha\,|\psi(\alpha)|^2 \equiv \int_{\mathbb{S}^1} \mathrm{d}\alpha\,|\psi(\alpha)|^2,$$

[1] Sums of angles are always to be understood $\mod 2\pi$, i.e., $\theta + \theta' \simeq (\theta + \theta') \mod 2\pi$.

and equipped with the scalar product

$$\langle \phi|\psi\rangle = \int_{\mathbb{S}^1} \mathrm{d}\alpha\, \overline{\psi(\alpha)}\, \phi(\alpha).$$

Given a real number $a \neq 0$, the action of the unitary irreducible representations of E(2) on $L^2(\mathbb{S}^1, \mathrm{d}\alpha)$ is realized as

$$L^2(\mathbb{S}^1, \mathrm{d}\alpha) \ni \psi(\alpha) \mapsto (U_a(\mathbf{r},\theta)\psi)(\alpha) = e^{\mathrm{i}a(r_1 \cos\alpha + r_2 \sin\alpha)}\psi(\alpha-\theta).$$

The cotangent bundle $T^*\mathbb{S}^1 \simeq (\mathbb{R}^2 \rtimes \mathrm{SO}(2))/H \simeq \mathbb{R}\times\mathbb{S}^1$ is viewed as the classical phase space for a particle moving on a circle. Choose H as

$$H \equiv H_{\hat{\mathbf{c}}} = \left\{(\mathbf{x},0) \in \mathrm{E}(2)\,|\,\hat{\mathbf{c}}\cdot\mathbf{x} = 0,\ \hat{\mathbf{c}} \in \mathbb{R}^2,\ \|\hat{\mathbf{c}}\| = 1,\ \text{fixed}\right\}.$$

The bundle $T^*\mathbb{S}^1$ carries canonical coordinates $(p,q) \in \mathbb{R}\times\mathbb{S}^1$ and the invariant measure is $\mathrm{d}p\,\mathrm{d}q \equiv \mathrm{d}p\wedge\mathrm{d}q$. Coordinates (p,q) are mapped to E(2) through a general section σ as $\mathbb{R}\times\mathbb{S}^1 \ni (p,q) \mapsto \sigma(p,q) = (\mathbf{f}(p,q),q) \in \mathrm{E}(2)$ where $\mathbf{f}(p,q)$ is a function to be determined. We then have:

Theorem 1. *Given a unit vector $\hat{\mathbf{c}} \in \mathbb{R}^2$ there exists a family of* affine *sections $\sigma : \mathbb{R}\times\mathbb{S}^1 \to E(2)$ defined as $\sigma(p,q) = (\mathcal{R}(q)(\boldsymbol{\kappa}p+\boldsymbol{\lambda}),q)$ where $\boldsymbol{\kappa},\boldsymbol{\lambda} \in \mathbb{R}^2$ are constant vectors, and $\hat{\mathbf{c}}\cdot\boldsymbol{\kappa} \neq 0$. The action of $E(2)$ on its left coset, through $(\mathbf{r},\theta)\sigma(p,q) = \sigma(p',q')(\mathbf{x},0)$, $\hat{\mathbf{c}}\cdot\mathbf{x} = 0$, is given by*

$$p' = p + \frac{1}{\hat{\mathbf{c}}\cdot\boldsymbol{\kappa}}\,\mathcal{R}(q+\theta)\hat{\mathbf{c}}\cdot\mathbf{r} \quad \textit{and} \quad q' = q+\theta.$$

This action is canonical, $\mathrm{d}p'\wedge\mathrm{d}q' = \mathrm{d}p\wedge\mathrm{d}q$.

Definition 2. With our choice of section, the UIR of E(2) with $a = 1$, and a choice of fiducial vector $\eta \in L^2(\mathbb{S}^1,\mathrm{d}\alpha)$, we define $|\eta_{p,q}\rangle = U(\sigma(p,q))|\eta\rangle$, i.e., $\eta_{p,q}(\alpha) = e^{\mathrm{i}[\mathcal{R}(q-\alpha)(\boldsymbol{\kappa}p+\boldsymbol{\lambda})]_1}\eta(\alpha-q)$ with

$$\boldsymbol{\kappa} = \kappa\begin{pmatrix}\cos\gamma\\ \sin\gamma\end{pmatrix} \quad\text{and}\quad \boldsymbol{\lambda} = \lambda\begin{pmatrix}\cos\zeta\\ \sin\zeta\end{pmatrix}.$$

Theorem 3. *The vectors $\eta_{p,q}$ form a family of coherent states for E(2) which resolves the identity on $L^2(\mathbb{S}^1,\mathrm{d}\alpha)$,*

$$I = \int_{\mathbb{R}\times\mathbb{S}^1} \frac{\mathrm{d}p\,\mathrm{d}q}{c_\eta}\,|\,\eta_{p,q}\,\rangle\langle\,\eta_{p,q}\,|,$$

if $\eta(\alpha)$ is admissible, *i.e.,* $\operatorname{supp}\eta \in (\gamma-\pi,\gamma) \mod 2\pi$ *and*

$$0 < c_\eta := \frac{2\pi}{\kappa}\int_{\mathbb{S}^1} \frac{|\eta(q)|^2}{\sin(\gamma-q)}\,\mathrm{d}q < \infty.$$

3. Quantization map and its covariance

With an admissible η, the quantization of a classical observable $f(p,q)$ is defined as the linear map

$$f \mapsto A_f^\sigma = \int_{\mathbb{R}\times\mathbb{S}^1} \frac{\mathrm{d}p\,\mathrm{d}q}{c_\eta} f(p,q)\,|\,\eta_{p,q}\rangle\langle\eta_{p,q}\,|.$$

To establish covariance of the quantization map, note that $\sigma_g : \mathrm{E}(2)/H_{\hat{\mathbf{c}}} \to \mathrm{E}(2)$ is a covariant translate of σ under $g = (\mathbf{r},\theta) \in \mathrm{E}(2)$,

$$\sigma_g(p,q) = g\sigma(g^{-1}(p,q)) = \sigma(p,q)h(g,g^{-1}(p,q)),$$

where the cocycle $h(g,(p,q))$ belongs to $H_{\hat{\mathbf{c}}}$. That is,

$$\sigma_g(p,q) = (\mathcal{R}(q)(\kappa p + \lambda), q)\left(\mathcal{R}(-q)\mathbf{r} - \kappa\,\frac{\mathcal{R}(q)\hat{\mathbf{c}}\cdot\mathbf{r}}{\hat{\mathbf{c}}\cdot\kappa}, 0\right).$$

Then,

$$U(g)A_f^\sigma U(g)^\dagger = A^{\sigma_g}_{\mathcal{U}_l(g)f}, \quad A_f^{\sigma_g} := \int_{\mathbb{R}\times\mathbb{S}^1} \frac{\mathrm{d}p\,\mathrm{d}q}{c_\eta} f(p,q)\,|\,U(\sigma_g(p,q)\eta\,\rangle\langle\,U(\sigma_g(p,q)\eta\,|\,,$$

with $\mathcal{U}_l(g)f(p,q) = f\left(g^{-1}(p,q)\right)$. The section itself is invariant under pure rotations $g = (\mathbf{0},\theta)$. The operator A_f^σ acts on the Hilbert space $L^2(\mathbb{S}^1,\mathrm{d}\alpha)$ as the integral operator with kernel

$$\mathcal{A}_f(\alpha,\alpha') = \frac{1}{c_\eta}\int_{\mathbb{S}^1} \mathrm{d}q\,\eta(\alpha-q)\overline{\eta(\alpha'-q)}e^{2\mathrm{i}\lambda S_\varsigma(\alpha,\alpha',q)}\int_{-\infty}^{+\infty}\mathrm{d}p\,e^{\mathrm{i}2\kappa S_\gamma(\alpha,\alpha',q)p}\,f(p,q)\,.$$

4. Some examples and classical limit

For the quantization of a function of the coordinate q, we have

Proposition 4. *For $f(p,q) = u(q)$ with $u(q+2\pi) = u(q)$, A_u is the multiplication operator $(A_u\psi)(\alpha) = (E_{\eta;\gamma} * u)(\alpha)\,\psi(\alpha)$ where the periodic convolution product on the circle is defined by*

$$(E_{\eta;\gamma} * u)(\alpha) = \int_{\alpha-\gamma}^{\alpha+\pi-\gamma} \mathrm{d}q\,E_{\eta;\gamma}(\alpha-q)\,u(q).$$

Here $E_{\eta;\gamma}(\alpha)$ is the positive 2π-periodic function

$$E_{\eta;\gamma}(\alpha) := \frac{2\pi}{\kappa c_\eta}\frac{|\eta(\alpha)|^2}{\sin(\gamma-\alpha)} \quad \textit{with} \quad \mathrm{supp}\,E_{\eta;\gamma} \subset (\gamma-\pi,\gamma).$$

For $0 \le \gamma < \pi$, $E_{\eta;\gamma}$ is a probability distribution on $\mathbb{S}^1$.

Proposition 5. *If the 2π-periodic function u is bounded on a period interval, then the 2π-periodic convolution $E_{\eta;\gamma} * u$ is bounded and continuous.*

The quantization is expected to regularize the original $u(q)$ (depending on the regularity of η). Note the translation covariance mod 2π, $U(\mathbf{0},\theta)A^{\sigma}_{f}U(\mathbf{0},\theta)^{\dagger} = A^{\sigma}_{f(\cdot-\theta)}$ which conveys to the quantum description the transition map from one chart to another one on the circle. Simple trigonometric functions become multiplication operators, as is the case with many other approaches, up to the presence of a multiplicative constant.

For real η, one has $(A_p\psi)(\alpha) = \left(-\mathrm{i}b\frac{\partial}{\partial\alpha} - \lambda a\right)\psi(\alpha)$, where a and b are constant. We note that, given η, one can choose the parameter κ such that $b = 1$ in order to get, up to the addition of an irrelevant constant, the familiar angular momentum operator $-\mathrm{i}\partial/\partial\alpha$, with spectrum $n \in \mathbb{Z}$. It is also interesting to note the role played by the parameter λ. It introduces a kind of gauge freedom, and since it is a free parameter, it can be chosen to be 0.

The semi-classical phase space portrait provided by the covariant or lower symbol

$$\check{f}(p,q) = \langle\,\eta_{p,q}\,|A_f|\,\eta_{p,q}\,\rangle = \int_{\mathbb{R}\times\mathbb{S}^1} \frac{\mathrm{d}p'\mathrm{d}q'}{c_\eta} f\,(p',q')\,|\langle\eta_{p',q'}|\eta_{p,q}\rangle|^2 \qquad (1)$$

of the operator A_f completes the quantization map $f \mapsto A_f$. It is the local average value of the original $f(q',p')$ with respect to the probability distribution $(q',p') \mapsto |\langle q,p|q',p'\rangle|^2$, i.e., the modulus squared of the overlap between two CS, on the phase space equipped with the measure $\mathrm{d}p'\mathrm{d}q'/c_\eta$.

The lower symbol for $f(p,q) = u(q)$ is $\check{u}(q) = \left[\,|\widetilde{\eta}|^2 * (E_{\eta;\gamma} * u)\,\right](q)$, where $\widetilde{\eta}(\alpha) = \eta(-\alpha)$. This convolution is expected to regularize the original $u(q)$. The lower symbol for $f(q,p) = p$ (with η real) is $\check{p} = cp + \lambda d$, where c and d are constants. With a fiducial vector η on the circle regularizing the Dirac delta, it is expected that within the framework of semi-classical analysis that $\check{f}$ approaches f as η becomes more localized.

Theorem 6. *The classical limit is obtained by considering the large κ limit (or, equivalently, $\hbar \to 0$) while imposing the condition $|\eta(\alpha)|^2 \to \delta(\alpha - \gamma + \pi/2)$ on the choice of fiducial vector η. In this case, the asymptotic expression for large κ is $\check{f}(q,p) = f(q,p) + o(1/\kappa)$.*

This is proved by applying the stationary phase approximation to the general expression for the lower symbol (1).

5. The Angle Operator

Quantization of the 2π-periodic and discontinuous angle function $a(\alpha) = \alpha$ for $\alpha \in [0, 2\pi)$ yields the multiplication operator $(A_a\psi)(\alpha) = (E_{\eta;\gamma} * a)\,(\alpha)\,\psi(\alpha)$. In the simplified case where $\lambda = 0$ and $\gamma = \pi/2$, one has $\operatorname{supp}\eta \subset (-\pi/2, \pi/2) \mod 2\pi$. We choose as fiducial vectors the family of 2π-periodic smooth even functions with support $[-\epsilon, \epsilon] \subset (-\pi/2, \pi/2) \mod 2\pi$, parametrized by $s > 0$ and $0 < \epsilon < \pi/2$,

$$\eta(\alpha) \equiv \eta^{(s,\epsilon)}(\alpha) = \frac{1}{\sqrt{\epsilon e_{2s}}}\, \omega_s\left(\frac{\alpha}{\epsilon}\right) \quad \text{where} \quad e_s := \int_{-1}^{1} \mathrm{d}x\, \omega_s(x).$$

Here $\omega_s(x)$ are the familiar smooth and compactly supported test functions for distributions $\omega_s(x) = \exp\left(-\dfrac{s}{1-x^2}\right)\chi_{(-1,1)}(x)$. As a matter of fact, the family of the squares of these functions form a Dirac delta sequence with respect to each parameter, $\left(\eta^{(s,\epsilon)}\right)^2(\alpha) \to \delta(\alpha)$ as $\epsilon \to 0$ or as $s \to \infty$. In Figure 1 we present plots of the spectrum of the angle operator and of its lower symbol.

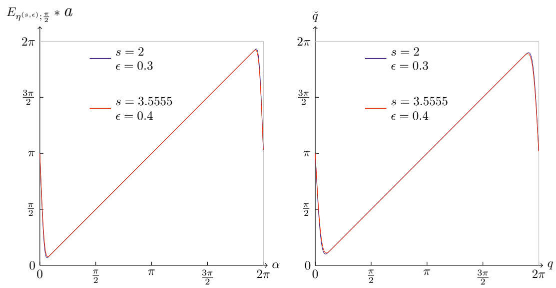

FIGURE 1. (Left) The multiplication angle operator $\left(E_{\eta^{(s,\epsilon)};\frac{\pi}{2}} * a\right)(\alpha)$ coincides with the angle function a inside $[\epsilon, 2\pi-\epsilon]$, while outside that interval the function $F_{\eta^{(s,\epsilon)};\frac{\pi}{2}}(\alpha)$ regularizes a; (Right) A plot of the lower symbol $\check{q}(q)$ of the angle operator A_a.

For $\lambda = 0$ and $\psi(\alpha) \in L^2(\mathbb{S}^1, \mathrm{d}\alpha)$, one has

$$\big([A_p, A_a]\psi\big)(\alpha) = -\mathrm{i}\mathsf{c}\,(1 - 2\pi E_{\eta;\gamma}(\alpha))\,\psi(\alpha)\,. \tag{2}$$

The constant factor c can be made equal to 1. This is a regularisation of the Dirac comb $(-\mathrm{i} + \mathrm{i}2\pi\delta(\alpha))$, which is recovered in the limit of the sequence of fiducial vectors with the choice $\kappa = 1$ (for further details, see [1]).

One main issue regarding the definition of an acceptable angle operator concerns the quantum angular dispersion versus the quantum angular momentum one. The Heisenberg inequality computed with the coherent states $\eta_{p,q}$ is

$$\Delta A_p\, \Delta A_a \geqslant \frac{1}{2}\mathsf{c}\left|1 - 2\pi\left(\left(\widetilde{\eta}^{(s,\epsilon)}\right)^2 * E_{\eta^{(s,\epsilon)};\frac{\pi}{2}}\right)(q)\right|\,.$$

In Figure 2 we show saturation plots for the uncertainty relations.

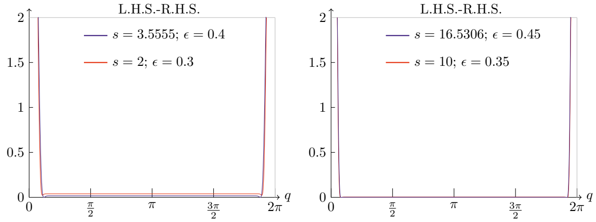

FIGURE 2. Differences between the left- and right-hand side of the uncertainty relation with respect to the coherent state $|\,\eta_{p,q}^{(s,\epsilon)}\,\rangle$ for various $\tau = s/\epsilon^2$. The state $|\,\eta_{p,q}^{(s,\epsilon)}\,\rangle$ saturates the Heisenberg inequality for large $\tau = s/\epsilon^2$

6. Conclusion

The angle function $a(\alpha) = \alpha$ for $\alpha \in [0, 2\pi)$ is mapped to a self-adjoint multiplication angle operator A_a. The continuous spectrum of A_a is $[\pi - m(s,\epsilon), \pi + m(s,\epsilon)]$, where $m(s,\epsilon) \to \pi$ as $\epsilon \to 0$ or $s \to \infty$. Thus systems like the classical pendulum or the torsion spring (where the angular motion is restricted) can be quantized without major issues. We present the (non-canonical) commutation rule between the angle and momentum operators, as well as an expression for the uncertainty relation between them. We also show how the classical limit can be obtained. However, a thorough study of the semi-classical limit, like the link between Poisson brackets and commutators is part of a future program.

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On Deformation Quantization using Super Twistorial Double Fibration

Yuji Hirota, Naoya Miyazaki and Tadashi Taniguchi

Dedicated to Professor Daniel Sternheimer in celebration of his 80th birthday

Abstract. We study deformation quantization for a complex supermanifold. Taking up a super twistor space whose body is a Calabi–Yau manifold concretely, we construct a double fibration and demonstrate that a certain super Calabi–Yau twistor space is deformation quantizable via the double fibration.

Mathematics Subject Classification (2000). Primary 53D55; Secondary 32C11, 32L25.

Keywords. Deformation quantization, complex supermanifold, twistor theory, Calabi–Yau manifold.

1. Introduction

Deformation quantization is one approach to obtaining a quantum system from a given classical system. The origin of it goes back to the work of F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer [2], based on Weyl's quantization [14]. The study of deformation quantization has been developed until now by many researchers: the existence of a star product for any symplectic manifold was shown by M. De Wilde and P. B. Lecomte [4]. Proofs of the existence, using symplectic connections and Čech cohomology, were given by B. V. Fedosov [5] and H. Omori, Y. Maeda and A. Yoshioka [11]. The existence and classification of the star product for any Poisson manifold was established by M. Kontsevich [6]. Recently, deformation quantization for a holomorphic Poisson manifold is discussed by N. Miyazaki [10] in algebraic/analytic categories with a certain condition.

We are mainly concerned with a supergeometry and quantization of a supermanifold. In the article, we focus on a super Calabi–Yau twistor space which is referred by M. Wolf [15] and discuss deformation quantization of it. One of representative examples of super Calabi–Yau twistor spaces is a complex projective supermanifold $\mathbb{P}^{\ |n}$. After giving a short overview on complex supermanifolds and fundamental examples of super Calabi–Yau twistor spaces, we define deformation quantization for Poisson supermanifolds and construct a double fibration from a complex supermanifold $\mathbb{C}^{4|8} \times \mathbb{P}^1$ to $\mathcal{P}^{\ |4}$ and $\mathbb{C}^{4|8}$. By using the double fibration, we show that the supermanifold $\mathcal{P}^{\ |4}$ is deformation quantizable.

2. Super Calabi–Yau twistor spaces

We begin this section by recalling the definition of a supermanifold. We suppose that the reader has fundamental knowledge concerning superalgebras. For the further information for supermanifolds, we refer the reader to [7, 8, 9].

Let m, n be natural numbers. Let $\theta^1, \ldots, \theta^n$ be generators of a Grassmann algebra, which satisfy the relations $\theta^i\theta^j + \theta^j\theta^i = 0$ for each $i, j = 1, \ldots, n$. Define a sheaf $\mathcal{O}_{m|n}$ of commutative superalgebras on $\mathbb{C}^m$ as

$$U \longmapsto \mathcal{O}_{m|n}(U) := \mathcal{O}(U)[\theta^1, \ldots, \theta^n]$$

for any open set $U \subset \mathbb{C}^m$. Here, $\mathcal{O}(U)[\theta^1, \ldots, \theta^n]$ is a Grassmann algebra with $\theta^1, \ldots, \theta^n$ over $\mathcal{O}(U)$, the space of holomorphic functions on U. We denote a ringed space $(\mathbb{C}^m,\ \mathcal{O}_{m|n})$ by $\mathbb{C}^{m|n}$.

Definition 1. A complex supermanifold of dimension $(m|n)$ is a pair $(M, \mathcal{A}_M)$ of an m-dimensional complex manifold M and a sheaf $\mathcal{A}_M$ of supercommutative rings on M such that every point in M has an open neighborhood which is isomorphic to some open subset of $\mathbb{C}^{m|n}$ as a ringed space.

When there is no confusion, we sometimes denote by $M^{m|n}$ a complex supermanifold $(M, \mathcal{A}_M)$ of dimension $(m|n)$, and often express a point of a supermanifold as

$$(z \,|\, \theta) := (z_1, z_2, \ldots, z_m \,|\, \theta^1, \theta^2, \ldots, \theta^n).$$

using the local coordinates $z_1, z_2, \ldots, z_n$ of M. By the definition, local sections f in $\mathcal{A}_M(U)$ are given in the form

$$f(z \,|\, \theta) = \sum_{k=1}^{n} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} f_{i_1 i_2 \ldots i_k}(z)\, \theta^{i_1}\theta^{i_2} \cdots \theta^{i_k}, \tag{1}$$

where $f_{i_1 i_2 \ldots i_k}(z)$ are holomorphic functions on an open set U of M.

Obviously, $\mathbb{C}^{m|n}$ is a complex supermanifold. There are some complex supermanifolds other than it as we shall show as below. A complex supermanifold on which we focus in the article is the one called a super Calabi–Yau twistor space.

Definition 2. A complex supermanifold $M^{\ |n}$ of dimension $(3 \,|\, n)$ is called a super Calabi–Yau twistor space if

(1) $p : M \longrightarrow \mathbb{P}^1$ is a holomorphic vector bundle over a complex projective space $\mathbb{P}^1$;
(2) M has a family of holomorphic section of p whose normal bundle $\mathcal{N}$ is isomorphic to

$$\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \big(\mathbb{C}^n \otimes_{\mathbb{C}} \Pi\mathcal{O}_{\mathbb{P}^1}(1)\big),$$

where the letter Π stands for the functor of changing the parity and $\mathcal{O}_{\mathbb{P}^1}(1)$ denotes the sheaf of sections of the dual of the tautological line bundle over $\mathbb{P}^1$;
(3) M is a Calabi–Yau manifold in the sense that M is a compact Kähler manifold whose first Chern class is zero.

If those two conditions other than (3) in Definition 2 are satisfied, a complex supermanifold $M^{\ |n}$ is called a super twistor space (see [12]). We introduce two examples of a super Calabi–Yau twistor space in what follows below.

Example. Let us consider a ringed space

$$\mathbb{P}^{\ |n} := \Big(\mathbb{P}\ ,\ \bigwedge^{\bullet}\big(\mathbb{C}^n \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}}\ (-1)\big)\Big),$$

where $\mathcal{O}_{\mathbb{P}}\ (-1)$ denotes the sheaf of sections of the tautological line bundle over a 3-dimensional complex projective space $\mathbb{P}$. It is a complex supermanifold of dimension $(3\,|\,n)$, called a $(3\,|\,n)$-dimensional projective superspace. For the sake of simplicity, we use the notation $\mathcal{O}_{\mathbb{P}\ |n}$ for the structure sheaf $\bigwedge^{\bullet}\big(\mathbb{C}^n \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}}\ (-1)\big)$. A local section of $\mathcal{O}_{\mathbb{P}\ |n}$ is represented as in (1) via a degree $(-k)$ homogeneous element $f_{i_1 i_2 \dots i_k}(z)$. As a result, the complex super projective space $\mathbb{P}^{\ |n}$ is a super twistor space.

Furthermore, the first Chern class is calculated to be

$$c_1\big(\mathbb{P}^{\ |n}\big) = c_1\big(\mathcal{O}_{\mathbb{P}\ |n} \otimes \mathbb{C}^4\big) - c_1\big(\mathcal{O}_{\mathbb{P}\ |n} \otimes \mathbb{C}^n\big) = (\quad - n)\,x$$

where $x = c_1\big(\mathcal{O}_{\mathbb{P}}\ (1) \otimes \mathcal{O}_{\mathbb{P}\ |n}\big)$. From this formula, the super twistor space $\mathbb{P}^{\ |n}$ is a super Calabi–Yau twistor space when $n = \ $.

Example. Let $k_1, \dots, k_{m+1} \quad \mathbb{Z}$. We denote by $\mathbb{WP}^m[k_1, k_2, \dots, k_{m+1}]$ a quotient space $\big(\mathbb{C}^{m+1} \setminus \{\mathbf{0}\}\big)/\mathbb{C}^\times$ by a $\mathbb{C}^\times$-action

$$t \cdot (z_1, \dots, z_{m+1}) := (t^{k_1} z_1, \dots, t^{k_{m+1}} z_{m+1}), \qquad t \quad \mathbb{C}^\times,$$

and by $\mathcal{O}_{\mathbb{WP}^m}(d)$ the sheaf of germs of holomorphic homogeneous functions on $\mathbb{WP}^m[k_1, k_2, \dots, k_{m+1}]$ of degree $d \quad \mathbb{Z}$. Then, a ringed space

$$\big(\mathbb{WP}^m[k_1, k_2, \dots, k_{m+1}],\ \mathcal{O}_{\mathbb{WP}^{m|n}}\big)$$

is a complex supermanifold, where

$$\mathcal{O}_{\mathbb{WP}^{m|n}} := \bigwedge^{\bullet}\big(\mathcal{O}_{\mathbb{WP}^m}(-\ell_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{WP}^m}(-\ell_n)\big).$$

This is called the weighted projective superspace (see [15, p. 40]), denoted by $\mathbb{WP}^{m|n}[k_1, k_2, \ldots, k_{m+1} \,|\, \ell_1, \ell_2, \ldots, \ell_n]$. It can be checked that the first Chern class is given by

$$c_1\Big(\mathbb{WP}^{m|n}[k_1, k_2, \ldots, k_{m+1} \,|\, \ell_1, \ell_2, \ldots, \ell_n]\Big) = \Big(\sum_{i=1}^{m+1} k_i - \sum_{j=1}^{n} l_j\Big)\, x, \tag{2}$$

where $x = c_1\big(\mathcal{O}_{\mathbb{WP}^m}(1) \otimes \mathcal{O}_{\mathbb{WP}^{m|n}}\big)$.

Let us consider the case where all of those integers are 1. We put

$$\mathbb{P}^{m|n} = \mathbb{WP}^{m|n}[1, 1, \ldots, 1 \,|\, 1, 1, \ldots, 1]$$

and define $\mathcal{P}^{\;\;|n} := \mathbb{P}^{\;\;|n} \setminus \mathbb{P}^{1|n}$. Remark that its body manifold is just $\mathbb{P}\;\;\setminus \mathbb{P}^1$, which is a $\mathbb{C}^2$-bundle over $\mathbb{P}^1$. It is shown that

$$\mathcal{P}^{\;\;|n} \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \bigoplus^{n} \Pi\mathcal{O}_{\mathbb{P}^1}(1).$$

This implies that $\mathcal{P}^{\;\;|n}$ is a super twistor space. From formula (2), it follows that $c_1(\mathbb{WP}^{\;\;|n})$ vanishes if $n = \;\;$. Therefore, $\mathcal{P}^{\;\;|4}$ is a super Calabi–Yau twistor space.

3. Deformation Quantization of Super Poisson algebras

3.1. Super Poisson algebras

Let $(M, \mathcal{A}_M)$ be a complex supermanifold of dimension $(m|n)$. For an open subset $U \subset M$, denote by $\Lambda^2(U, \mathcal{A}_M)$ the set of $\mathcal{A}_M(U)$-valued bilinear maps on $\Omega^1(U, \mathcal{A}_M)$ satisfying, for all $\alpha, \beta \;\; \Omega^1(U, \mathcal{A}_M)$ and $f \;\; \mathcal{A}_M(U)$,

$$\pi(\alpha, \beta) = (-1)^{1+|\alpha||\beta|}\, \pi(\beta, \alpha)$$

and

$$\pi(\alpha f, \beta) = (-1)^{|f||\beta|} \pi(\alpha, \beta) f.$$

Here, $\Omega^1(U, \mathcal{A}_M)$ denotes the right $\mathcal{A}_M(U)$-module of super differential 1-forms on U (see [7]). An element $\pi \;\; \Lambda^2(U, \mathcal{A}_M)$ is called a super Poisson structure if it satisfies $[\pi, \pi]_{\mathrm{SN}} = 0$, where $[\cdot, \cdot]_{\mathrm{SN}}$ stands for the super Schouten bracket (see [1]). It is verified that a super Poisson structure π on $M^{m|n}$ yields a super Poisson algebra in a similar way as an ordinary Poisson structure does a Poisson algebra.

A super Poisson algebra is a super vector space $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ equipped with two multiplications $(f, g) \mapsto fg$ and $(f, g) \mapsto \{f, g\}$ which satisfy

(1) $\{f, g\} = -(-1)^{|f||g|}\{g, f\}$;
(2) $(-1)^{|f||h|}\{f, \{g, h\}\} + (-1)^{|g||f|}\{g, \{h, f\}\} + (-1)^{|h||g|}\{h, \{f, g\}\} = 0$;
(3) $\{f, gh\} = \{f, g\}h + (-1)^{|f||g|} g\{f, h\}$

for any $f, g, h \;\; \mathcal{A}$. Here, $|f|$ denotes a parity for the homogeneous element f. The bracket $\{\cdot, \cdot\}$ is called a super Poisson bracket. A super Poisson bracket is said to be even if $|\{f, g\}| = |f| + |g|$, while it is said to be odd if $|\{f, g\}| = |f| + |g| + 1$. A supermanifold $(M, \mathcal{A}_M)$ whose structure sheaf $\mathcal{A}_M$ is a sheaf of a super Poisson algebra is called a Poisson supermanifold. For further discussion on Poisson supermanifolds, refer to [3].

Example. The complex supermanifold $\mathbb{C}^{m|n}$ is a symplectic supermanifold by a even 2-form

$$\omega_{\mathbb{C}^{m|n}} := \frac{\sqrt{-1}}{2}\sum_{i=1}^{m} dz_i \wedge d\bar{z}_i + \frac{1}{2}\sum_{j=1}^{n}(d\theta_j)^2$$

with respect to the standard coordinates $(z_1, \ldots, z_m \,|\, \theta_1, \ldots, \theta_n)$. In a similar manner to an ordinary symplectic manifold, for any $f \quad \mathcal{O}_{m|n}(\mathbb{C}^m)$, there is a super vector field ξ_f such that $\omega_{\mathbb{C}^{m|n}}(\xi_f, \cdot) = \mathrm{d}f(\cdot)$. Using those vector fields, we define a bracket $\{\cdot, \cdot\}$ as

$$\{f,\, g\} = \omega_{\mathbb{C}^{m|n}}(\xi_f,\, \xi_g)$$

for $f, g \quad \mathcal{O}_{m|n}(\mathbb{C}^m)$. The bracket makes a Poisson supermanifold of $\mathbb{C}^{m|n}$.

Given a super Poisson algebra $\mathcal{A}$, we denote by $\mathcal{A}[[\hbar]]$ the space of all formal power series $\sum_{=0}^{\infty} f \;\hbar$ in a parameter $\hbar$, where $f \quad \mathcal{A}$. We define deformation quantization for $\mathcal{A}$ as follows:

Definition 3. A super Poisson algebra $(\mathcal{A}, \{\cdot, \cdot\})$ is said to be deformation quantizable if there exists a product

$$* : \mathcal{A}[[\hbar]] \times \mathcal{A}[[\hbar]] \longrightarrow \mathcal{A}[[\hbar]]$$

satisfying the following conditions:

(1) $*$ is bilinear and $\hbar$-linear;
(2) $f * (g * h) = (f * g) * h$ for any $f, g, h \quad \mathcal{A}[[\hbar]]$;
(3) If $f * g = f \cdot g + \sum_{=1}^{\infty} \pi \;(f, g)\hbar$, then $\pi_1(f, g) = \frac{1}{2}\{f, g\}$.

The product $*$ is called the star product or $*$-product (see [2, 5, 6, 11, 16]). Using $*$, one can get a commutator by putting

$$[f,\, g]_* := f * g - (-1)^{|f||g|} g * f$$

for any $f, g \quad \mathcal{A}$. We say that a Poisson supermanifold $(M, \mathcal{A}_M)$ is deformation quantizable when the sheaf of super Poisson algebra $\mathcal{A}_M$ is deformation quantizable.

3.2. Main results

Let us consider a complex supermanifold $\mathbb{C}^{4|2n}$. We write x^{ij} $(i, j = 1, 2)$ for the standard coordinates $(z_1, \ldots, z_4)$ of $\mathbb{C}^4$ and does $\theta^{i\alpha}$ $(\alpha = 1, 2, \ldots, n)$ for the odd coordinates $(\theta_1, \ldots, \theta_{2n})$ like matrices. Choose a super Poisson structure ϖ on $\mathbb{C}^{4|2n}$ so that the matrix for ϖ is represented in the form

$$E := \begin{pmatrix} \varpi^{ij,k\ell} & \mathbf{O}_{4,2n} \\ \mathbf{O}_{2n,4} & \varpi^{i\alpha,k\beta} \end{pmatrix}, \tag{3}$$

where $\varpi^{ij,k\ell}$ is a $\quad\times\quad$-matrix from the super Poisson structure restricted to $\mathbb{C}^{4|0}$, and $\varpi^{i\alpha,k\beta}$ a $2n \times 2n$-matrix from the super Poisson structure restricted to $\mathbb{C}^{0|2n}$ $(i, j, k, \ell = 1, 2,\ \alpha, \beta = 1, 2, \ldots, n)$. We define a product $*$ on $\mathcal{O}_{4|2n}(U)$ for each open subset U of $\mathbb{C}^4$ as

$$(f * g)(x \mid \theta) := f(x \mid \theta) \exp\left[\frac{\hbar}{2} \sum \frac{\overleftarrow{\partial}}{\partial X^A} E \frac{\overrightarrow{\partial}}{\partial X^B}\right] g(x \mid \theta) \tag{4}$$

for any superfunction $f, g \quad \mathcal{O}_{4|2n}(U)$. Here, $\dfrac{\partial}{\partial X^A}$ stands for a family of superderivations with $(x^{ij} \mid \theta^{i\alpha})$ and $\dfrac{\partial}{\partial X^B}$ does for the one with $(x^{k\ell} \mid \theta^{k\beta})$.

Proposition 4. *A product by expanding* (4) *linearly to* $\mathcal{O}_{4|2n}[[\hbar]]$ *is the star product and its commutator* $[\cdot, \cdot]_*$ *satisfies the following*:

$$[x^{ij}, x^{k\ell}]_* = \hbar\, \varpi^{ij,k\ell}, \quad [x^{ij}, \theta^{k\alpha}]_* = 0, \quad [\theta^{i\alpha}, \theta^{j\beta}]_* = \hbar\, \varpi^{i\alpha,j\beta}.$$

for any $i, j, k, \ell = 1, 2$ *and* $\alpha, \beta = 1, 2, \ldots, n$.

Consider the case of $n = \quad$. We construct a double fibration [13]

$$\mathcal{P}^{\ |4} \xleftarrow[\rho]{} \mathbb{C}^{4|8} \times \mathbb{P}^1 \xrightarrow[\sigma]{} \mathbb{C}^{4|8} \tag{5}$$

by putting

$$\rho : \mathbb{C}^{4|8} \times \mathbb{P}^1 \longrightarrow \mathcal{P}^{\ |4}, \quad (y^{ij}, [\boldsymbol{\lambda}] \mid \theta^{i\alpha}) \longmapsto [y^{ij}\lambda_j : \lambda_i \mid \theta^{i\alpha}\lambda_i]$$

and

$$\sigma : \mathbb{C}^{4|8} \times \mathbb{P}^1 \longrightarrow \mathbb{C}^{4|8}, \quad (y^{ij}, [\boldsymbol{\lambda}] \mid \theta^{i\alpha}) \longmapsto (y^{ij} \mid \theta^{i\alpha}),$$

where $y^{ij} := x^{ij} - \sum_{k=1}^{2} \theta^{ik}\theta^{k,j+2}$ for $i, j = 1, 2$ and where $[\boldsymbol{\lambda}] = [\lambda_1 : \lambda_2]$ are homogeneous coordinates of $\mathbb{P}^1$. We remark that the suffix $\alpha \quad \mathbb{N}$ ranges from 1 to 4 and $y^{ij}\lambda_j$ means to be summed over j. The double fibration (5) brings a super Poisson structure on the super Calabi–Yau twistor space $\mathcal{P}^{\ |4}$ from a Poisson supermanifold $\mathbb{C}^{4|8}$ and proves $\mathcal{P}^{\ |4}$ to be deformation quantizable.

Theorem 5. $\mathcal{P}^{\ |4}$ *is deformation quantizable.*

Proof. Take a super Poisson structure ϖ on $\mathbb{C}^{4|8}$ like (3). Pulling ϖ back on $\mathbb{C}^{4|8} \times \mathbb{P}^1$ with σ, and moreover pushing it forward to $\mathcal{P}^{\ |4}$ with ρ, we get a tensor field π on $\mathcal{P}^{\ |4}$ written locally in the form

$$\pi = \lambda_1\lambda_2 \frac{\partial}{\partial \zeta^1} \wedge \frac{\partial}{\partial \zeta^2} + \frac{1}{2}\left({\lambda_1}^2 + {\lambda_2}^2\right) \sum_{\alpha=1}^{4} \frac{\partial}{\partial \xi^\alpha} \wedge \frac{\partial}{\partial \xi^\alpha},$$

where $\zeta^i = y^{ij}\lambda_j, \xi^\alpha = \theta^{i\alpha}\lambda_i$ for each i, α. We can check $[\pi, \pi]_{\mathrm{SN}} = 0$, which implies that $\mathcal{P}^{\ |4}$ is a Poisson supermanifold.

Like (4), we define a product $\star$ as

$$(f \star g)(\zeta \mid \xi) := f(\zeta \,|\, \xi) \exp\left[\frac{\hbar}{2} \sum \frac{\overleftarrow{\partial}}{\partial Z^A}\, \pi\, \frac{\overrightarrow{\partial}}{\partial Z^B}\right] g(\zeta \,|\, \xi), \tag{6}$$

where $\dfrac{\partial}{\partial Z^{\bullet}}$ denotes a family of superderivations with the newly re-arranging coordinates $(\zeta^i, \lambda_j \,|\, \xi^\alpha)$ of $\mathcal{P}^{\ |4}$. Similarly to Proposition 4, the product $\star$ extended linearly to $\mathcal{A}_{\mathcal{P}^{\ |4}}[[\hbar]]$ proves to be the star product satisfying

$$[\zeta^1, \zeta^2]_\star = 2\hbar\, \lambda_1 \lambda_2, \qquad [\xi^\alpha, \xi^\alpha]_\star = \hbar\, ({\lambda_1}^2 + {\lambda_2}^2) \quad (\alpha = 1, 2, 3, \ ),$$

where $[\cdot, \cdot]_\star$ stands for a commutator given by

$$[f,\, g]_\star := f \star g - (-1)^{|f||g|} g \star f$$

for any $f, g \quad \mathcal{A}_{\mathcal{P}^{\ |4}}[[\hbar]]$. Therefore, $\mathcal{P}^{\ |4}$ is deformation quantizable. □

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Deformation Quantization of Commutative Families and Vector Fields

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Abstract. We describe a series of cohomological obstructions for the deformation of involutive families of functions on a Poisson manifold and for the deformation of Poisson vector fields acting on it.

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1. Introduction: commutative families and obstructions. Examples

This article contains a review of the various cohomological obstructions for the deformation quantization of involutive families on a Poisson manifold M (i.e. collections of functions on (M, π) that commute with respect to the Poisson bracket) and of the Lie algebra actions on M.

More accurately, let M be a smooth manifold and π be a Poisson bivector on it, i.e. the Schouten bracket of π with itself vanish. By Kontsevich's theorem there always exists an associative $*$-product $*$ on $C^{\infty}(M)[[\hbar]]$ (here $\hbar$ is a formal variable) such that

$$f * g = fg + \frac{\hbar}{2}\{f, g\} + \sum_{k \geq 2} \hbar^k B_k(f, g) \tag{1}$$

for any $f, g \quad C^{\infty}(M)$, where B_k are suitable bidifferential operators and $\{\cdot, \cdot\}$ is the Poisson bracket, induced by π; for technical reasons it is often convenient to denote the bidifferential operator determined by Poisson bracket as $2B_1$, so that $B_1(f, g) = \frac{1}{2}\{f, g\}$ and denote the product of functions by B_0.

So we shall denote by $\mathcal{A}$ the associative algebra $(C^{\infty}(M)[[\hbar]], *)$; it is called a deformation quantization of M. Let $C \subset C^{\infty}(M)$ be an algebra of functions on M, that commute with respect to the Poisson bracket, i.e. $\{f, g\} = 0$ for all $f, g \quad C$. The main question, we deal with is *for what C there exists a commutative subalgebra $\widehat{C} \subset \mathcal{A}$, $\widehat{C} \equiv C \mod \hbar$?*

One of the traditional devices, used for this kind of problems is to look for the cohomology classes (in suitable cohomology theories) that should vanish in the case there exists an algebra $\widehat{C}$ with the desired properties. This was the main idea of the paper [1]: in this paper the problem of finding the commutative subalgebra in $\mathcal{A}$, corresponding to $C \subset C^\infty(M)$ freely generated by functions $f_1, \dots, f_n$ was reduced to vanishing of a series of classes in the second degree cohomology of certain complex, denoted by $C^{\cdot}_f$ in this paper (a similar construction was used in [2]).

It is easy to see that in fact, $C^{\cdot}_f$ from the cited paper is just the Chevalley complex, computing the Lie algebra cohomology of the Abelian Lie algebra of Hamiltonian fields, generated by $f_1, \dots, f_n$ with coefficients in $C^\infty(M)$. This cohomology can be rather big, so the criterion is rather difficult to implement. In the cited paper of Garay and van Straten it was also shown that the obstruction classes (or "anomalies" as they are called in the cited paper) vanish, if $M = \mathbb{R}^{2n}$ (where n is the number of functions in the commutative family) with the standard symplectic structure, the functions $f_1, \dots, f_n$ are functionally independent and the corresponding cohomology spaces are torsion-free, as modules over the polynomial algebra $\mathbb{C}[t_1, \dots, t_n]$, where the action of t_i is given by multiplication with f_i. However, the statement can be true even under much milder assumptions. Here's a simple example:

Example 1. Consider the real Lie algebra $\mathfrak{so}$; we can assume that its generators are x, y, z with commutator relations

$$[x, y] = z, \ [y, z] = x, \ [z, x] = y.$$

Let $M = \mathfrak{so}^*$ with the standard (linear) Poisson structure. Observe that x, y and z can be regarded as linear functions on M so that $S(\mathfrak{so}\ ) = \mathbb{C}[x, y, z]$ give a Poisson subalgebra of $C^\infty(M)$. In this case we can restrict the quantization procedure to $S(\mathfrak{so}\ )$; the resulting "polynomial" quantized algebra $\mathcal{A}_0$ can be related to the universal enveloping algebra $U(\mathfrak{so}\ )$. Thus the question of finding commutative deformations of involutive families is closely related with the problem of describing the commutative subalgebras of $U(\mathfrak{so}\ )$. Consider for example the following pair of commuting functions on M: $f = x^2 + y^2$, $g = z^k$ for arbitrary natural exponential k (their commutation follows from the fact that $x^2 + y^2 + z^2$ is a Casimir function). It is not difficult to find the corresponding commutative subalgebra in $U(\mathfrak{so}\ )$: it is enough to consider x, y and z as the corresponding generators of the enveloping algebra.

Now fix $k = 1$; identifying M with $\mathbb{R}$ in an evident way we see that the Hamiltonian vector fields, generated by f and g correspond to rotation around the z axis (the first field in fact is equal to the second one multiplied by z). The dimension of the commutative Lie algebra $\mathfrak{t}_2$ generated by f and g is equal to 2, thus the second Lie algebra cohomology of $\mathfrak{t}_2$ is equal to cokernel of the differential. If we pass to the cylindrical coordinates r, φ, z in $\mathbb{R}$, then $\mathfrak{t}_2$ is generated (up to constant factors) by the fields $rz\partial_\varphi$ and $r\partial_\varphi$ respectively; thus the cohomology is equal to the quotient of $C^\infty(\mathbb{R}\ )$ by the image of $r\partial_\varphi$. In cylindrical coordinates smooth polynomial functions on $\mathbb{R}$ are those, which depend on r^2; on the other hand it follows from the computations of S^1 cohomology that the only function of φ, that is not in the image of ∂_φ is constant. Thus we conclude that this cohomology can be identified with polynomials of $r^2 = f$ and $z = g$, in particular this space is torsion-free as the module over the subalgebra, generated by f and g. So the criterion of Garay and van Straten holds too. This might indicate that this statement has a wider scope of application, than what is mentioned in [1].

Before we end this section, let us consider one more simple example:

Example 2. Let $n = 2$. Recall that Hamiltonian vector field X_f, generated by a function f is determined by equality

$$X_f(g) = \{f, g\}$$

for all $g \quad C^\infty(M)$. Let us assume that the first of two functions in the involutive family f_1 is such that the corresponding Hamiltonian field is nowhere vanishing. In this case there is a very nice way to choose the deformation quantization of the involutive family f_1, f_2, $\{f_1, f_2\} = 0$ (below we shall denote $f_1 = f, f_2 = g$ for the sake of brevity). Namely, *we shall put* $\widehat{f} = f \quad C^\infty(M) \subset \mathcal{A}$ *and look for* $\widehat{g} = g + \hbar g_1 + \hbar^2 g_2 + \cdots \quad \mathcal{A}$ *such that* $[\widehat{f}, \widehat{g}] = 0$.

By induction this will give us a series of differential equations:

$$\{f, g_p\} = \sum_{k=1}^{p} \left(B_{k+1}(f, g_{p-k}) - B_{k+1}(g_{p-k}, f)\right), \tag{2}$$

where B_k are the bidifferential operators from formula (1) and we set $g_0 = g$. When $g_0 = g, g_1, \ldots, g_{p-1}$ are known, the right hand side of this formula is fixed, so we have an ordinary differential equation for g_p. Locally this equation always has a solution: it is enough to choose local coordinate system $x^1, \ldots, x^m,\ m = \dim M$ on M in which the Hamiltonian field X_f of f will be equal to ∂_1; then the equation (2) takes the form

$$\partial_1 g_p = G_p,$$

for some function G_p of $x^1, \ldots, x^m$ and can be solved by simple integration. However, globally these solutions need not "fit" into a single function on M. In this case (although this construction is a bit redundant) one can consider the Chevalley complex of the 1-dimensional Lie algebra, generated by X_f with coefficients in $C^\infty(M)$; then the existence of global solutions of (2) is equivalent to the vanishing of the cohomology class, determined by the right hand side of this equation. However this cohomology space is quite big so this criterion has very little practical meaning.

Another way to "visualize" the cohomological obstruction in this situation is by considering the Čech complex on M corresponding to the cover of M by coordinate charts with the property $\partial_1 = X_f$; we shall take coefficients of this complex in the sheaf of locally-defined functions, "killed" by X_f. Then the difference of local solutions of (2) on the intersections of two such open neighborhoods will give a cocycle in this complex, so that the cohomology class will vanish iff there exists a global solution of this equation.

In fact, it is easy to see that the problem we consider here always has a local solution in symplectic case. Namely let M^{2n} be symplectic manifold and $f_1, \dots, f_n$ an involutive family. Assume that the Hamiltonian fields of these functions are linearly independent in some point. Then by Darboux theorem one can choose local coordinates $(p_1, \dots, p_n, q_1, \dots, q_n)$ in M so that $f_i = p_i,\ i = 1, \dots, n$ and the Poisson bivector has form

$$\pi = \sum_{i=1}^{n} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i}.$$

As one knows from the uniqueness property of Kontsevich quantization, in this coordinate chart Kontsevich's formula is equivalent by a gauge transformation to Moyal quantization, in particular $[p_i, p_j] = 0$ (where the commutator is taken with respect to the $*$-product). Once again if we assume that the family $f_1, \dots, f_n$ is everywhere functionally independent, the problem can be reduced to the vanishing condition for certain cohomology classes C_k in the degree 1 Čech cohomology of M with coefficients in the sheaf of (local) gauge transformations of the $*$-product, preserving the Darboux coordinates.

1.1. Agreements and notation

In the rest of this paper we shall discuss many other cohomological obstructions for the positive solution of the problem, stated here. These obstructions will take values in various cohomology theories, first of all in Hochschild cohomology of the algebra of (smooth) functions on M and in the Poisson cohomology of (M, π); we shall not discuss the construction of these cohomology spaces, this would have taken too much time and effort: an interested reader should consult for example the Loday's book [3], or the book [4]. The main theorem, we need about Hochschild cohomology is the Hochschild–Kostant–Rosenberg's theorem, relating the Hochschild cohomology with the space of polyvector fields on M; besides this, we shall use the identification of the Lichnerowicz–Poisson cohomology of symplectic manifolds and their de Rham cohomology. Another cohomology theory that will make often appearances in the paper is the Chevalley's Lie algebra cohomology; reasonable expositions of this construction can be found in a great variety of books on homological algebra and Lie groups, for example see [5]. We shall also assume that the reader is familiar with Kontsevich's approach to deformation quantization (see [6], which contains a nice introduction to the subject); in particular we shall rather freely operate with the notions of differential graded Lie algebras, Gerstenhaber and Schouten brackets, L_∞-algebras and L_∞ morphisms, quasi-isomorphisms, etc.

All this can be found in numerous papers on the subject; some notation and formulas can be found e.g. in the paper [7].

The ground field we consider here is always $\mathbb{R}$, although we allow smooth functions with complex values. In particular, all the Lie algebras in this text are supposed to be finite-dimensional and over real numbers. All the manifolds are smooth, but not necessarily compact.

2. Poisson fields: three-step process

It is clear from the examples in the previous section that the answer to the original question depends in a great extend on the global structure of the Hamiltonian vector fields, their singular sets and trajectories. Therefore it is the study of the Hamiltonian field action that we shall consider from now on. In fact, it is more convenient to consider a less restrictive setting, namely the action of Poisson fields, (recall that a vector field is called Poisson if the Lie derivative of π with respect to this field vanishes). Below we shall denote the space of all Poisson vector fields on (M, π) by $Vect_\pi(M)$; clearly, it is a Lie subalgebra in the algebra of vector fields on M and one can show that Hamiltonian fields is an ideal in $Vect_\pi(M)$. Observe that from algebraic point of view, vector fields on a manifold M are just derivations of the algebra of smooth functions on it. Recall that derivation is a linear map ξ of an A algebra into itself, such that $\xi(ab) = \xi(a)b + a\xi(b)$ for all $a, b \quad A$.

Summing up previous observations, we come up with the following problem: let $\mathfrak{g}$ be a Lie algebra, acting on M by Poisson vector fields. It means that there is a Lie algebra representation $\rho : \mathfrak{g} \to Vect_\pi(M)$:

$$[\rho(X), \rho(Y)] = \rho([X, Y]), \text{ for all } X, Y \quad \mathfrak{g}.$$

(In the case of the Hamiltonian fields generated by an involutive family, the commutators should vanish). Then the problem we consider is to *find the extension of the representation ρ to $\widehat{\rho} : \mathfrak{g} \to \mathrm{Der}(\mathcal{A})$*, i.e. find a linear map $\widehat{\rho}$ from $\mathfrak{g}$ to the space of all $\hbar$-linear derivations of $\mathcal{A}$ such that

$$[\widehat{\rho}(X), \widehat{\rho}(Y)] = \widehat{\rho}([X, Y]), \text{ for all } X, Y \quad \mathfrak{g}$$

and $\widehat{\rho} \equiv \rho \mod \hbar$.

As we have explained above, the problem of deforming a Lie algebra representation is closely related with the original one. In fact we can try to solve the problem of deformation of involutive families in three stages, each step being a separate problem of independent interest:

- Find the representation $\widehat{\rho} : \mathfrak{g} \to \mathrm{Der}(\mathcal{A})$, extending the action of the Lie algebra $\mathfrak{g}$ on M (in particular, the action of commutative algebra of Hamiltonian fields);
- Let $\widehat{X} \quad \mathrm{Der}(\mathcal{A})$ be a derivation such that $\widehat{X} = X_f + \hbar X_1 + \hbar^2 X_2 + \cdots$ where $X_i,\ i \geq 1$ are some differential operators and X_f is the Hamiltonian field of some smooth function; then the question is whether $\widehat{X}$ is equal to an

inner derivation, i.e. $\widehat{X}(g) = [\widehat{f}, g]$ for some $\widehat{f}$ and all g in $\mathcal{A}$. Observe that the series for $\widehat{f}$ in this context should begin with $\hbar^{-1} f$, since the Poisson bivector appears with coefficient $\hbar$ in the series for $*$-product. To avoid the negative powers of $\hbar$ below we shall rather consider the series $\widehat{X}$ beginning with $\hbar X_f$.

- Finally, suppose we have a Lie algebra representation $\widehat{\rho} : \mathfrak{g} \to \mathrm{Der}(\mathcal{A})$ and for every $X \in \mathfrak{g}$ we have an element $\widehat{f}_X \in \mathcal{A}$ such that $\widehat{\rho}(X)(g) = [\widehat{f}_X, g]$ for all $g \in \mathcal{A}$; then, one can ask, if the functions $\widehat{f}_X$ can be chosen so that the map $X \to \widehat{f}_X$ is a representation of $\mathfrak{g}$ (in particular, $[\widehat{f}_X, \widehat{f}_Y] = 0$ for all $X, Y \in \mathfrak{g}$, if $\mathfrak{g}$ is commutative).

We shall discuss the first two problems in this list in the remaining sections of our paper. Let us now briefly describe the cohomology behind the last one.

To this end we suppose that $\widehat{\rho}$ is a representation of Lie algebras and that we have an element $\widehat{f}_X \in \mathcal{A}$ chosen for all $X \in \mathfrak{g}$. By choosing a basis in $\mathfrak{g}$ we can assume that this choice is in fact given by a linear map $f : \mathfrak{g} \to \mathcal{A},\ X \mapsto \widehat{f}_X$. Consider the following formula:

$$\mathrm{ad}_{\widehat{f}_{[X,Y]}} = \widehat{\rho}([X, Y]) = [\widehat{\rho}(X), \widehat{\rho}(Y)] = [\mathrm{ad}_{\widehat{f}_X}, \mathrm{ad}_{\widehat{f}_Y}] = \mathrm{ad}_{[\widehat{f}_X, \widehat{f}_Y]}\,.$$

Here ad_f denotes the inner derivative in $\mathcal{A}$, generated by $f \in \mathcal{A}$: $\mathrm{ad}_f(g) = [f, g]$ and we use the standard properties of the commutator. It follows from this equality that inner derivation, induced by $\widehat{f}_{[X,Y]}$ and the commutator $[\widehat{f}_X, \widehat{f}_Y]$ is the same. Thus the difference

$$h(X, Y) = \widehat{f}_{[X,Y]} - [\widehat{f}_X, \widehat{f}_Y]$$

is in the center $Z(\mathcal{A})$ of the algebra $\mathcal{A}$. The map $h : \mathfrak{g} \times \mathfrak{g} \to Z(\mathcal{A})$ is clearly antisymmetric, thus it can be regarded as a map $h : \wedge^2 \mathfrak{g} \to Z(\mathcal{A})$. On the other hand since derivations of an algebra preserve its center, we have a Lie algebra action of $\mathfrak{g}$ on $Z(\mathcal{A})$ and can consider the Chevalley complex of $\mathfrak{g}$ with coefficients in $Z(\mathcal{A})$; so $h \in C^2(\mathfrak{g}, Z(\mathcal{A}))$. Then a straightforward calculation shows that

$$\begin{aligned} dh(X, Y, Z) &= \widehat{\rho}(X)(h(Y, Z)) + \text{cyclic permutations of } X,\ Y,\ Z \\ &\qquad + h([X, Y], Z) + \text{cyclic permutations of } X,\ Y,\ Z = 0 \end{aligned}$$

since the first three terms vanish identically, because $h(X, Y)$ is in center and $\widehat{\rho}$ acts by inner representations, and the second line vanishes due to the properties of the commutators.

We need to find $\widehat{f}_X$ and $\widehat{f}_Y$ such that $h(X, Y) \equiv 0$; on the other hand we cannot change the inner derivations, determined by these functions, since the representation $\widehat{\rho}$ is fixed. So we can only add elements from the center $Z(\mathcal{A})$ to $\widehat{f}_X$; if $c : \mathfrak{g} \to Z(\mathcal{A})$ is this correction term, then h will be changed by $dc(X, Y) = \widehat{\rho}(X)c(Y) - \widehat{\rho}(Y)c(X) - c([X, Y]) = c([Y, X])$. Thus we see that the following statement holds:

Proposition 1. *Suppose we have a Lie algebra representation $\widehat{\rho} : \mathfrak{g} \to \mathrm{Der}(\mathcal{A})$ such that for every X in $\mathfrak{g}$ one can find an element $\widehat{f}_X \in \mathcal{A}$ such that $\widehat{\rho}(X) = \mathrm{ad}_{\widehat{f}_X}$.*

Then there exists a Lie algebra representation $J : \mathfrak{g} \to \mathcal{A}$ (i.e. a linear map J for which $[J(X), J(Y)] = J([X, Y])$) such that $\widehat{\rho} = ad \circ J$ iff the class of h in $H^2(\mathfrak{g}, Z(\mathcal{A}))$ is trivial.

Remark 2. It is known that the center of $\mathcal{A}$ is usually isomorphic to the center $Z_\pi(M)$ of the original Poisson algebra $(C^\infty(M), \{,\})$ (for instance, in the case, when he quantization is given by Kontsevich's formula, this follows from the existence of the associated Kontsevich's "tangent map", in other cases one can refer to the uniqueness of the quantization). So the homology $H^2(\mathfrak{g}, Z(\mathcal{A}))$ can be replaced by $H^2(\mathfrak{g}, Z_\pi(M))$ where we let $\mathfrak{g}$ act on $Z_\pi(M)$ in the usual way. It is an interesting question, how one can express the image of the class of h in $H^2(\mathfrak{g}, Z_\pi(M))$ under this identification.

3. Equivariant quantization and homological obstructions

One of the possible ways to solve the problem of finding Lie algebra action on $\mathcal{A}$ is by assuming that the quantization/star-product in $\mathcal{A}$ is $\mathfrak{g}$-equivariant (this is sometimes also called $\mathfrak{g}$-invariant). In this section we shall first discuss the question when there exist equivariant quantization and then describe cohomological obstructions one encounters when passing from arbitrary derivation of $\mathcal{A}$ to an inner one.

3.1. Equivariant quantization

In the present section we shall discuss a simple way (probably the most naive one) how one can solve the problem contained in first step of the three-stage process, described above. For the sake of brevity we shall often omit the representation $\rho : \mathfrak{g} \to Vect_\pi(M)$ from our notation here.

Recall that a deformation quantization is called $\mathfrak{g}$-equivariant, if the corresponding $*$-product commutes with the action of $\mathfrak{g}$:

$$\mathcal{L}_X(f * g) = (\mathcal{L}_X f) * g + f * (\mathcal{L}_X g), \tag{3}$$

where we let the vector fields act on the elements of $C^\infty(M)[[\hbar]]$ by setting $\mathcal{L}_X(\hbar) = 0$. We shall begin with the following well-known result (see [8] and [9] for a generalized version of this statement, which includes the Poisson structures):

Proposition 3. *Let (M, ω) be a symplectic manifold, on which a Lie algebra $\mathfrak{g}$ acts by Poisson vector fields. Then there exists a **$\mathfrak{g}$-equivariant deformation quantization** of M if there exists a $\mathfrak{g}$-equivariant symplectic connection on TM.*

Here we call a connection ∇ on M symplectic, if the covariant derivative of the symplectic form ω with respect to ∇ vanishes. Similarly, the connection is called $\mathfrak{g}$-*equivariant* (or $\mathfrak{g}$-*invariant*) if it commutes with the action of $\mathfrak{g}$, i.e. if the following commutator is trivial:

$$[\mathcal{L}_X, \nabla] = \mathcal{L}_X \circ \nabla - \nabla \circ \mathcal{L}_X = 0$$

for all X $\mathfrak{g}$ (passing to efficient representation, if necessary, we may identify the Lie algebra and the set of Poisson fields, obtained by its representation).

Observe that Proposition 3 gives only a *sufficient* condition for the existence of equivariant deformation quantization, i.e. for the existence of star-product, verifying the condition (3). It is not clear, what would be a necessary and sufficient condition in this case. However, if the condition of Proposition 3 is verified, we can use the tautological representation of $\mathfrak{g}$ in the construction we described above, that is we can just put $\widehat{\rho} \equiv \rho$. This construction can be used in some important cases. For instance, if the action of $\mathfrak{g}$ can be integrated to a symplectic action of a compact Lie group G (for instance, if all the Liouville tori of an integrable system are compact and all their periods are commeasurable, i.e. in the so-called "resonance case") then we can always obtain an equivariant connection on M just by averaging over G the translations of an arbitrary symplectic connection on M.

So let us now discuss the possible cohomological nature of the condition in Proposition 3. Suppose that ∇ is an arbitrary symplectic connection (their existence is provided by the non-degeneracy of the symplectic form (see the book [10] for example). Of course, this connection need not be $\mathfrak{g}$-invariant, so its Lie derivative with respect to an element X $\mathfrak{g}$ is a non-trivial $(0,1)$-tensor field on M with values in the degree-preserving endomorphisms of the tensor fields on M; indeed for any vector field ξ $Vect(M)$, any tensor field η $T^{\otimes}M$ and any function f $C^{\infty}(M)$

$$\begin{aligned}
(\mathcal{L}_X\nabla)(\xi, f\eta) &= \mathcal{L}_X(\nabla_\xi(f\eta)) - \nabla_{\mathcal{L}_X\xi}(f\eta) - \nabla_\xi(\mathcal{L}_X(f\eta)) \\
&= [X, \nabla_\xi(f\eta)) - \nabla_{[X,\xi]}(f\eta) - \nabla_\xi([X, f\eta]) \\
&= [X, \xi(f)\eta] + [X, f\nabla_\xi\eta] - [X,\xi](f)\eta \\
&\quad - f\nabla_{[X,\xi]}\eta - \nabla_\xi(X(f)\eta) - \nabla_\xi(f[X,\eta]) \\
&= X(\xi(f))\eta + \xi(f)[X,\eta] + X(f)\nabla_\xi\eta + f[X,\nabla_\xi\eta] - [X,\xi](f)\eta \\
&\quad - f\nabla_{[X,\xi]}\eta - \xi(X(f))\eta - X(f)\nabla_\xi\eta - \xi(f)[X,\eta] - f\nabla_\xi([X,\eta]) \\
&= f([X,\nabla_\xi\eta] - \nabla_{[X,\xi]}\eta - \nabla_\xi([X,\eta])) = f(\mathcal{L}_X)(\xi,\eta)
\end{aligned}$$

and similarly $(\mathcal{L}_X\nabla)(f\xi,\eta) = f(\mathcal{L}_X\nabla)(\xi,\eta)$. Besides this, since ∇ is a symplectic connection, and $\mathfrak{g}$ acts by Poisson vector fields, $\mathcal{L}_X\nabla(\omega) = 0$. Let us denote the space of the endomorphism of tensor fields on M which send ω to 0 by $End_\omega(T^{\otimes}M)$, so the rule $X \mapsto \mathcal{L}_X\nabla$ gives us a linear map $a : \mathfrak{g} \to \Omega^1(M, End_\omega(T^{\otimes}M))$. The Lie algebra $\mathfrak{g}$ acts on $End_\omega(T^{\otimes}M)$ and on the space of 1-forms with values in it by Lie derivatives, so $\Omega^1(M, End_\omega(T^{\otimes}M))$ is a $\mathfrak{g}$-module and we can consider the Chevalley complex of $\mathfrak{g}$ with coefficients in this module. A simple computations shows that

$$da(X,Y) = \mathcal{L}_X(a(Y)) - \mathcal{L}_Y(a(X)) - a([X,Y]) = 0.$$

The following proposition is a straightforward consequence of the definitions and observations made above:

Proposition 4. *There exists a symplectic $\mathfrak{g}$-invariant connection on M iff the class of a in the Chevalley cohomology of $\mathfrak{g}$ with coefficients in $\Omega^1(M, End_\omega(T^{\otimes}M))$ vanishes.*

Of course, the space of all tensor endomorphisms is quite big and redundant, since both ∇ and $\mathcal{L}_X$ are derivations with respect to the tensor product, everything is determined by the induced maps $TM \to TM$. In this case the condition $(a(X))(\omega) = 0$ is replaced by the condition $\omega(a(X)(\xi), \eta) + \omega(\xi, a(X)\eta) = 0$ for all vector fields ξ, η and all $X \quad \mathfrak{g}$; one can rephrase it by saying that for all X, $a(X)$ is symplectic endomorphism of TM with respect to ω, or in matrix form

$$a(X)_i^j \omega_{jk} = -\omega_{ij} a(X)_k^j.$$

Thus, we can regard a as a map $\mathfrak{g} \to \Omega^1(M, \mathfrak{sp}(TM, \omega))$ (on the right we have the space of linear symplectomorphisms of TM with respect to the fibrewise symplectic structure, determined by ω). Thus we can rephrase Proposition 4:

Proposition 5. *There exists a symplectic $\mathfrak{g}$-invariant connection on M iff the class of a in the Chevalley cohomology of $\mathfrak{g}$ with coefficients in $\Omega^1(M, \mathfrak{sp}(TM, \omega))$ vanishes.*

One can call the class of *a the Atiyah class* of the symplectic Lie algebra action. It is possible to use it to define other cohomology classes; for instance, by taking its trace we obtain a class in $H^1(\mathfrak{g}, \Omega^1(M))$. More generally, by taking traces of its powers we obtain classes in $H^k(\mathfrak{g}, \Omega^k(M))$. It is clear, that these classes should vanish, if there exist equivariant symplectic connection, but not the otherwise.

3.2. Inductive constructions and obstructions: inner derivations

Let us now suppose that we are given a representation $\widehat{\rho}$ of Lie algebra $\mathfrak{g}$ in $\mathrm{Der}(\mathcal{A})$ (for instance, this is the case when the conditions of Proposition 3 hold). We are going to discuss the problem of finding the inner derivatives, corresponding to this representations in the case when the derivations in the image of $\widehat{\rho}$ are given by Hamiltonian fields modulo $\hbar$; i.e. suppose that for any $Y \quad \mathfrak{g}$ we have

$$\widehat{\rho}(Y) = X_{H_Y} + \hbar Y_1 + \hbar^2 Y_2 + \cdots, \tag{4}$$

where H_Y is a suitable Hamiltonian function, depending on $Y \quad \mathfrak{g}$ and Y_k, $k \geq 1$ are some differential operators on M. Then we are to find a linear map $\mathfrak{g} \to \mathcal{A}$, sending Y to some $\widehat{f_Y} \quad \mathcal{A}$ such that $\widehat{\rho}(Y) = \mathrm{ad}_{\widehat{f_Y}}$.

First of all observe that since the conditions we have to fulfill are linear, it is enough to consider the case when $\mathfrak{g}$ is one-dimensional: if we can find f_Y for all Y in a basis of $\mathfrak{g}$ then extending the map to the whole $\mathfrak{g}$ by linearity solves the problem. So let $\widehat{Y} = \widehat{\rho}(Y) \quad \mathrm{Der}(\mathcal{A})$ be some derivation of $\mathcal{A}$ given by the formula (4). Below we shall abbreviate X_{H_Y} by Y_0 and H_Y by f; so Y_0 is a Hamiltonian vector field: $Y_0(g) = \{f, g\}$. The condition that $\widehat{Y}$ is a derivation is equivalent to the following system of equations:

$$\delta(Y_n) + \sum_{k=1}^{n} [B_k, Y_{n-k}] = 0, \tag{5}$$

where δ is the Hochschild cohomology differential and the bracket on the right is the Gerstanhaber bracket (see [3, 7] for definitions); it is worth noting that $\delta(x) = [B_0, x]$ where B_0 is the product in $C^\infty(M)$. Our problem is to find an element $\widehat{f} \quad \mathcal{A}$ such that

$$\mathrm{ad}_{\widehat{f}} = \widehat{Y}. \tag{6}$$

Let us begin by recalling the following simple statement from the theory of Hochschild cohomology:

Proposition 6. *A Hochschild cochain $v \quad CH^1(\mathcal{A})$ is closed iff v is a differentiation of $\mathcal{A}$; it is exact iff this differentiation is inner.*

This statement gives a tautological answer to the question we consider here. However, this answer is rather difficult to use, since Hochschild homology of $\mathcal{A}$ is usually quite big. So in what follows we shall describe a more "hand-on" method.

To this end let us write $\widehat{f}$ as the formal sum $\widehat{f} = \sum_{k\in\mathbb{Z}} \hbar^k f_k,\ f_k \quad C^\infty(M)$, then equation (6) can be written down in the form of a series of equations

$$Y_n = \sum_{p+q=n} [B_p, f_q]. \tag{7}$$

It is necessary to allow the negative powers of $\hbar$ since the multiplication B_0 is commutative; alternatively and in a more convenient way, we could have replaced the formula (4) by

$$\widehat{Y}' = \hbar Y_0 + \hbar^2 Y_1 + \hbar\ \ Y_2 + \cdots. \tag{8}$$

Assuming this notation we can avoid the use of $\hbar^{-1}$; in this case we can rewrite equations (7) as follows:

$$\begin{aligned} Y_0 &= [B_1, f_0] = X_{f_0}, \\ Y_1 &= [B_2, f_0] + [B_1, f_1], \\ &\ \dots \\ Y_n &= \sum_{k=0}^{n} [B_{n-k+1}, f_k] \\ &\ \dots \end{aligned} \tag{9}$$

We can regard this as a series of equations on $f_k,\ k = 0, 1, 2, \dots$ It is clear that one should take $f_0 = f = H_Y$ in order to satisfy the first equality. Let us now consider the second equation. We rewrite it as

$$[B_1, f_1] = Y_1 - [B_2, f_0] \tag{10}$$

and apply the Hochschild differential δ to both sides of it: on the left we shall have

$$[\delta(B_1), f_1] - [B_1, \delta(f_1)] = 0$$

since $\delta \equiv 0$ on $C^\infty(M)$, $\delta(B_1) = 0$. On the right we have

$$\delta Y_1 = -[B_1, Y_0] - [\delta(B_2), f_0] + [B_2, \delta(f_0)] = \frac{1}{2}[[B_1, B_1], f_0] = 0.$$

The first term on the right is obtained from (5); it vanishes since Y_0 is a Hamiltonian vector field. The equality $\delta B_2 = -\frac{1}{2}[B_1, B_1]$ is a part of the associativity relation of the $*$-product; then we use the Jacobi identity for the Gerstenhaber bracket and the earlier obtained equalities $[B_1, f_0] = Y_0,\ [B_1, Y_0] = 0$.

Thus we can pass to the Hochschild cohomology here. Let us now apply the Lichnerowicz–Poisson differential to the class, determined by $Y_1 - [B_2, f_0]$; we are going to show that it is equal to 0. To this end it is enough to consider the Gerstenhaber bracket of this expression with B_1 and consider the result modulo the image of δ. We compute:

$$\begin{aligned}
[B_1, Y_1] - [B_1, [B_2, f_0]] &= [B_1, Y_1] - [[B_1, B_2], f_0] + [B_2, Y_0] \\
&= \delta Y_2 + [\delta B\ \ , f_0] \\
&= \delta(Y_2 + [B\ \ , f_0])
\end{aligned}$$

where we use the Jacobi identity, associativity relation for $*$ and equation (5) again. Thus the right hand side of (10) corresponds to a closed element in Lichnerowicz-Poisson complex; let us denote it by w_1. Now we can rewrite the equation (10) as

$$d_\pi f_1 = w_1.$$

And we come to conclusion that *there exists f_1 iff the class of w_1 in $H^1_\pi(M)$ is trivial.* The only thing that might need some clarification is that the vanishing property of the Poisson cohomology class only entails the existence of a solution of (10) up to a Hochschild coboundary; however since $C^\infty(M)$ is commutative algebra, all 1-dimensional coboundaries in the corresponding Hochschild cohomology complex are equal to 0.

Proceeding by induction, we assume that the equations for $f_0, f_1, \dots, f_{n-1}$ have been solved. Then we rewrite the corresponding equation in (9) in the form

$$[B_1, f_n] = Y_n - \sum_{k=0}^{n-1} [B_{n-k+1}, f_k].$$

As before we show that both sides of this equation are closed with respect to the Hochschild differential δ and that the element w_n in the Hochschild cohomology $HH^1(C^\infty(M)) = Vect(M)$, corresponding to the expression on the right hand side of this equation, is closed with respect to the Lichnerowicz's Poisson cohomology differential; we shall denote the corresponding class in $H^1_\pi(M)$ by the same symbol w_n. Thus we obtain the following

Proposition 7. *A differentiation $\widehat{Y}$ $\ \ \mathrm{Der}(\mathcal{A})$, given by* (8) *with $Y_0 = X_f$, is inner iff a certain series of cohomology classes $w_1, w_2, \dots, w_n, \dots$ $\ \ H^1_\pi(M)$ vanish.*

In general this condition is very hard to check, but in the case π is non degenerate, so that M is symplectic manifold, its Lichnerowicz's Poisson cohomology coincide with the usual de Rham cohomology. So in this case we have the following simple consequence:

Corollary 8. *If M is a symplectic 1-connected manifold, then every derivation $\widehat{Y}$ $\mathrm{Der}(\mathcal{A})$ given by (8) with $Y_0 = X_f$, is inner. In particular this is true, if $Y_0 = 0$.*

4. The representation of Lie algebras

In this section we are going to address the main part of our construction: given a Lie algebra $\mathfrak{g}$ acting by Poisson vector fields, find an extension of this action to $\mathcal{A}$. In Section 3.1 we considered a particular case when the action of $\mathfrak{g}$ could be extended to $\mathcal{A}$ in a straightforward way since the quantization is equivariant. In the present section we shall describe more general constructions.

4.1. Inductive construction and obstructions: Lie algebra representations

One of the most straightforward possible approaches to the problem of quantization of a Lie algebra action (i.e. of extending it from $C^\infty(M)$ to $\mathcal{A}$) is by constructing this extension inductively step by step with respect to the powers of $\hbar$. This approach has been earlier considered by the author (see [7, 11]) and is in effect a simple generalization of the approach used by Garay and van Straten (see [1] and [2]). Let us briefly describe the results; an interested reader can find some details in [7].

So let $Y = Y_0$ be a Poisson vector field; we are looking for an extension

$$\widehat{Y} = Y_0 + \hbar Y_1 + \hbar^2 Y_2 + \cdots$$

of Y to a derivation of $\mathcal{A}$. Writing down the Leibniz rule and expanding everything in the powers of $\hbar$ we obtain a system, similar to (5): in that place these equations were supposed to hold by the virtue of our assumption that $\widehat{Y}$ was a derivation; now we shall use them to find the missing terms Y_1, $Y_2, \ldots$

Reasoning inductively, we can assume that the elements $Y_1, \ldots, Y_{n-1}$ are given; then simple computations show that the element $\sum_{k=1}^{n}[B_k, Y_{n-k}]$ in this equation is closed with respect to the Hochschild differential and that the corresponding element in the Hochschild cohomology is closed with respect to the Lichnerowicz–Poisson differential. On the other hand, the first term in this sum is $[B_1, Y_{n-1}]$ where B_1 is the bidifferential operator, given by the bivector π (up to a constant multiple), so adding vector fields (regarded as Hochschild cocycles) to Y_{n-1} will not change the Poisson cohomology class of this sum. This gives us the following (see [7])

Proposition 9. *The element Y_n that solves equation (5) exists if the class v_n $H^2_\pi(M)$ (where $H^*_\pi(M)$ denotes the Poisson cohomology of M) vanishes.*

In particular, if M is a symplectic manifold, then we can identify H^2_π with the second de Rham cohomology of M; in particular, if $M = \mathbb{R}^{2n}$ with some symplectic structure, then the problem always has positive solution. Another case when this problem has positive solution is $M = \mathfrak{g}^*$ for a semisimple Lie algebra $\mathfrak{g}$: in this case $H^*_\pi(M)$ coincide with the Lie algebra cohomology of $\mathfrak{g}$ which is known to vanish in dimension 1 in many cases (see [12]).

Unfortunately, this construction only works for a single Poisson vector field, but does no good for a Lie algebra representation, since in this case there are two algebraic conditions that we should verify: the Leibniz rule and the condition that commutator of derivations is equal to the image of the commutator of the corresponding elements in $\mathfrak{g}$:

$$[\widehat{\rho}(X), \widehat{\rho}(Y)] = \widehat{\rho}([X, Y]), \text{ for all } X, Y \quad \mathfrak{g}. \tag{11}$$

Let us write $\widehat{\rho} : \mathfrak{g} \to \mathrm{Der}(\mathcal{A}) \subset End(\mathcal{A})$ as a formal power series

$$\widehat{\rho} = \rho_0 + \hbar\rho_1 + \hbar^2\rho_2 + \cdots ,$$

where ρ_0, ρ_1, $\rho_2, \ldots$ are linear maps $\mathfrak{g} \to End(\mathcal{A})$ ($End(\mathcal{A})$ denotes the space of $\hbar$-linear endomorphisms of $\mathcal{A}$ and $\rho_0 = \rho : \mathfrak{g} \to Vect_\pi(M)$ is the given representation of $\mathfrak{g}$ in Poisson vector fields on M). Since $\mathcal{A} = C^\infty(M)[[\hbar]]$ as linear space, we can assume that the elements of $End(\mathcal{A})$ are given by differential operators on $C^\infty(M)$, thus we have an isomorphism of $End(\mathcal{A})$ with the space of (local) Hochschild 1-cochains on $C^\infty(M)$, denoted by $C^1(C^\infty(M))$. If we consider the Chevalley (bi)complex $C^*(\mathfrak{g}, C^*(C^\infty(M)))$ (where one makes $\mathfrak{g}$ act on the Hochschild complex via the representation ρ and the second differential is given by the Hochschild coboundary map) then $\rho_k \quad C^1(\mathfrak{g}, C^1(C^\infty(M)))$ and the condition (11) turns into the following series of equations:

$$d_L(\rho_n) + \frac{1}{2}\sum_{k=1}^{n-1}[\rho_k, \rho_{n-k}] = 0. \tag{12}$$

Here d_L denotes the Chevalley differential on the bicomplex $C^*(\mathfrak{g}, C^*(C^\infty(M)))$ and we use the brackets to denote the following operation on $C^1(\mathfrak{g}, C^1(C^\infty(M)))$:

$$\begin{aligned} &[\cdot,\cdot] : C^1(\mathfrak{g}, C^1(C^\infty(M))) \otimes C^1(\mathfrak{g}, C^1(C^\infty(M))) \to C^2(\mathfrak{g}, C^1(C^\infty(M))),\\ &\quad [\varphi, \psi](X, Y) = [\varphi(X), \psi(Y)] + [\varphi(Y), \psi(Y)]. \end{aligned} \tag{13}$$

Here on the right-hand side we use $[\cdot,\cdot]$ to denote the Gerstenhaber brackets in $C^*(C^\infty(M))$. If we consider the B_k as elements in $C^0(\mathfrak{g}, C^2(C^\infty(M)))$, then equations (5) in this notation will take the form

$$\delta(\rho_n) + d_L B_n + \frac{1}{2}\sum_{k=1}^{n-1}[B_k, \rho_{n-k}] = 0. \tag{14}$$

Here we use the same notation for the bracket, which is similar to (13):

$$\begin{aligned} &[\cdot,\cdot] : C^0(\mathfrak{g}, C^2(C^\infty(M))) \otimes C^1(\mathfrak{g}, C^1(C^\infty(M))) \to C^1(\mathfrak{g}, C^2(C^\infty(M)))\\ &\quad [\alpha, \psi](X) = [\alpha, \psi(X)] \end{aligned} \tag{15}$$

(once again we use the Gerstenhaber bracket on the right-hand side and observe that $[B_n, \rho_0(X)] = d_L B_n(X)$). Recall now that the associativity condition for $*$ can be written down in the form

$$\delta B_n + \frac{1}{2}\sum_{k=1}^{n-1}[B_k, B_{n-k}] = 0, \tag{16}$$

where $[\cdot,\cdot]$ denotes the Gerstenhaber bracket, which coincides with the natural operation on $C^0(\mathfrak{g}, C^2(C^\infty(M)))$ extending the brackets (13) and (15).

Moreover, observe that our equations do not involve any terms of the form $\rho_k^2(X,Y) \quad C^2(\mathfrak{g}, C^0(C^\infty(M)))$: in theory, these terms could have appeared in equation (12), so that it would look as

$$d_L(\rho_n) + \frac{1}{2}\sum_{k=1}^{n-1}[\rho_k, \rho_{n-k}] + \frac{1}{2}\sum_{k=1}^{n-1}[B_k, \rho_{n-k}^2] = 0, \tag{17}$$

where the last brackets denote the evident extension of the (13) and (15) and Gerstenhaber brackets to the map $C^2(\mathfrak{g}, C^0(C^\infty(M))) \otimes C^0(\mathfrak{g}, C^2(C^\infty(M))) \to C^2(\mathfrak{g}, C^1(C^\infty(M)))$. Equation (17) corresponds to the situation, where the representation of $\mathfrak{g}$ in $\mathrm{Der}(\mathcal{A})$ is given modulo the inner derivations, i.e. the following equation holds

$$[\widehat{\rho}(X), \widehat{\rho}(Y)] - \widehat{\rho}([X,Y]) = \mathrm{ad}_{\widehat{\rho}^2(X,Y)} \tag{18}$$

for some $\widehat{\rho}^2 = \rho_0^2 + \hbar\rho_1^2 + \cdots,\ \rho_k^2 \quad C^2(\mathfrak{g}, C^0(C^\infty(M)))$. Below we shall discuss this formula in the context of L_∞-maps.

Recall, that a degree 1 element R in a differential graded Lie algebra is called Maurer–Cartan element, if it verifies the equation

$$dR + \frac{1}{2}[R,R] = 0. \tag{19}$$

Thus, summing up, we see from equations (12), (14) and (16) that the problem we consider here is in some sense equivalent to the search of *Maurer–Cartan element* in the Lie algebra $\bar{C}^*(\mathfrak{g}, C^*(C^\infty(M)[[\hbar]]))[1]$ (here [1] denotes the shift of dimension, so that the elements ρ_k and B_k become of degree 1 and $\bar{C}^*$ denotes the subcomplex, spanned by the elements of positive degrees in $\mathfrak{g}$), where we endow this complex with the natural Lie algebra structure, first, extending the Gerstenhaber bracket on the Hochschild complex and the brackets from formulas (13), (15) to the whole $C^*(\mathfrak{g}, C^*(C^\infty(M)))$ by the following formula: for any $\varphi \quad C^p(\mathfrak{g}, C^l(C^\infty(M)))$ and $\psi \quad C^q(\mathfrak{g}, C^m(C^\infty(M)))$

$$\begin{aligned}[\varphi,\psi](X_1,\ldots,X_{p+q}) = \sum_{\sigma\in Sh(p,q)} &(-1)^{1+(l-1)|\sigma|}\\ &\times [\varphi(X_{\sigma(1)},\ldots,X_{\sigma(p)}), \psi(X_{\sigma(p+1)},\ldots,X_{\sigma(p+q)})],\end{aligned} \tag{20}$$

where the sum is taken over all (p,q) shuffles, $|\sigma|$ is the parity of the permutation σ and we use the Gerstenhaber brackets in the right hand side. We further extend this bracket by $\hbar$-linearity and restrict to the $\bar{C}^*$ subcomplex. We can now formulate the following proposition:

Proposition 10. *There exists a $*$-product in $C^\infty(M)[[\hbar]]$ beginning with the given Poisson structure π and a representation $\widehat{\rho}$ of the Lie algebra $\mathfrak{g}$ in the space of derivations* $\mathrm{Der}(\mathcal{A})$ *of the deformed algebra $\mathcal{A} = (C^\infty(M)[[\hbar]], *)$, equal modulo $\hbar$ to the given representation of $\mathfrak{g}$ in the space of Poisson vector fields $Vect_\pi(M)$, iff there exists a Maurer–Cratan element* $\Pi + \mathrm{P}$ *in $\bar{C}^*(\mathfrak{g}, C^*(C^\infty(M)[[\hbar]]))[1]$, equal modulo $\hbar$ to $\pi + \rho$.*

Remark 11. We said that the search of Maurer–Cartan element is only similar to the problem, because in the original problem we assumed that the elements $B_k,\ k = 1, 2, \ldots$ are fixed. Meanwhile in the present formulation of the problem we fix only the degree 1 part of the $*$-product, i.e. the operator B_1, determined by Poisson bracket.

On the other hand this liberty makes the problem closer to the classical problem of deformation quantization: compare the complex $\bar{C}^*(\mathfrak{g}, C^*(C^\infty(M)[[\hbar]]))[1]$ with $\bar{C}^*(\mathfrak{g}, \mathcal{T}^*(M)[[\hbar]])[1]$, where $\mathcal{T}^*(M)$ denotes the Lie algebra of polyvector fields on M and $\bar{C}^*$ as before denotes the subcomplex spanned by polyvectors of positive degrees; one can use the Schouten bracket on $\mathcal{T}^*(M)$ to get a Lie algebra structure on $C^*(\mathfrak{g}, \mathcal{T}^*(M)[[\hbar]])[1]$ and restrict it to $\bar{C}^*$. The evident generalization of the Hochschild–Kostant–Rosenberg map will then determine an isomorphism in cohomology and one can look for the L_∞-quasi-isomorphism of these algebras, extending the HKR map (observe that the absence of the degree 0 part does not spoil the HKR isomorphism, since the Hochschild coboundary is trivial in this degree).

It is easy to see that if it were not for the condition $\rho_k^2 = 0,\ k \geq 0$, the problem would have had a positive answer. Indeed the classical Kontsevich's construction allows a rather straightforward extension to a quasi-isomorphism of unrestricted complexes $C^*(\mathfrak{g}, \mathcal{T}^*(M)[[\hbar]])[1]$ and $C^*(\mathfrak{g}, C^*(C^\infty(M)[[\hbar]]))[1]$; thus the main problem is whether one can change this quasi-isomorphism so that the degree 0 part does not appear. Below we shall give a different construction, leading to an analogous result. Also observe that this problem is very close to the Garay and van Straten calculus of anomalies: in both case we have to change the map so as to eliminate the defect of the commutator. In effect, if we construct the correction terms, killing the element $\widehat{\rho}^2 = \sum_k \hbar^k \rho_k^2$ inductively (with respect to the powers of $\hbar$), then we shall obtain "anomaly" classes, similar to those from the paper [1] (a more detailed analysis of these constructions we postpone to a forthcoming work).

Also observe, that it follows from the definitions that the element $\widehat{\rho}^2$ is closed with respect both to Chevalley and Hochschild differentials, thus it represents a class in the cohomology of $C^*(\mathfrak{g}, C^*(C^\infty(M)[[\hbar]]))[1]$. It is not clear, how this class is related with the question we consider here, since the equations, concerned with its' "elimination" involve elements, containing B_k and ρ_k and not just generic elements of the DG Lie algebra.

4.2. Lie algebra representations: L_∞-morphisms

The construction of the derivation $\widehat{Y}$ from the previous section is quite nice; but in effect it is a little more than one needs: the obstructions of Proposition 9 show, when one can extend to a derivation a given "finite degree approximation" $\widehat{Y}^n = Y_0 + \hbar Y_1 + \hbar^2 Y_2 + \cdots + \hbar^n Y_n$, rather than to answer the question, if there exists some formal series with the necessary property, may be obtainable by some other, non-inductive process. In addition, we only allow changing the last chosen element at every step, so that the condition of this proposition is not necessary, but only sufficient (unlike the condition of Proposition 10).

It turns out that in the case when the $*$-product is given by some L_∞-morphism between the differential graded Lie algebras $\mathcal{T}^*(M)$ of polyvector fields on M (with zero differential and Schouten bracket) and $C^*(C^\infty(M))$ of (local) Hochschild cochains on $C^\infty(M)$ (with Hochschild boundary operator and Gerstenhaber brackets), there exists a canonical choice of the derivation $\widehat{Y}$, corresponding to a Poisson vector field Y. Moreover, there also exist a canonical way to find the "representation up to inner derivations", i.e. a pair of linear maps $\widehat{\rho}^1 = \widehat{\rho} : \mathfrak{g} \to \mathrm{Der}(\mathcal{A})$, $\widehat{\rho}^2 : \wedge^2\mathfrak{g} \to \mathcal{A}$ such that the equation (18) would hold.

To this end suppose that $\mathscr{U} = \{U_n\}$, $U_n : \wedge^n\mathcal{T}^*M \to C^*(C^\infty(M))$ is some L_∞-quasi-isomorphism, extending the Hochschild–Kostant–Rosenberg's map (for instance, Kontsevich's morphism). Suppose also that the coefficients of the $*$-product in $\mathcal{A}$ are given by the formula

$$B = \sum_{k\geq 1} \hbar^k B_k = \sum_{n\geq 1} \frac{\hbar^k}{k!} U_k\big(\underbrace{\pi, \dots, \pi}_{k \text{ times}}\big).$$

Then we shall put:

$$\widehat{\rho}^1(X) = \sum_{k\geq 0} \frac{\hbar^k}{k!} U_{k+1}\big(\rho(X), \underbrace{\pi, \dots, \pi}_{k \text{ times}}\big).$$

An easy computation with the definition of L_∞-morphisms shows that this formula determines a linear map $\widehat{\rho}^1 : \mathfrak{g} \to \mathrm{Der}(\mathcal{A})$. However the map $\widehat{\rho}^1$ fails to be a representation of the Lie algebras: the correction term is given by

$$\widehat{\rho}^2(X, Y) = \sum_{k\geq 0} \frac{\hbar^{k+1}}{k!} U_{k+2}\big(\rho(X), \rho(Y), \underbrace{\pi, \dots, \pi}_{k \text{ times}}\big),$$

so that we have, similarly to the equation (18)

$$[\widehat{\rho}^1(X), \widehat{\rho}^1(Y)] - \widehat{\rho}^1([X, Y]) = \mathrm{ad}_{\widehat{\rho}^2(X,Y)}\,.$$

Here, as before ad_x, $x \quad \mathcal{A}$ is the inner derivative of $\mathcal{A}$ with respect to the element x, i.e. $\mathrm{ad}_x(y) = [x, y] = x * y - y * x$.

In order to answer the question, when this kind of representation can be replaced with a usual one, consider the complex $CHE^{\cdot}(\mathfrak{g}, \mathcal{A})$ consisting of all linear maps from the exterior algebra of $\mathfrak{g}$ to the $\hbar$-linear Hochschild cohomology complex of $\mathcal{A}$, $CH^{\cdot}(\mathcal{A})$. We introduce the grading in $CHE^{\cdot}(\mathfrak{g}, \mathcal{A})$ by putting $\deg f = p + q$ for a map $f : \wedge^p(\mathfrak{g}) \to C^q(\mathcal{A})$; we further introduce the differential $d_{CHE} : CHE^n(\mathfrak{g}, \mathcal{A}) \to CHE^{n+1}(\mathfrak{g}, \mathcal{A})$: if $f = \{f_p\},\ p = 0, \ldots, n$, where f_p is a homogeneous map as described above (with $q = n - p$), then $d = d_{CHE}$ is given by the next formula (in which we omit the wedge signs)

$$\begin{aligned} df(X_1, \ldots, X_p) = \delta(f_p(X_1, \ldots, X_p)) & \\ - \sum_i (-1)^{i(n-p)} [\widehat{\rho}^1(X_i), f_{p-1}(X_1, \ldots, \widehat{X_i}, \ldots, X_p)] & \\ - \sum_{i<j} (-1)^{i+j} [\widehat{\rho}^2(X_i, X_j), f_{p-2}(X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_p)] & \\ + \sum_{i<j} (-1)^{i+j} f_{p-1}([X_i, X_j], X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_p), & \end{aligned}$$

where as usually δ is the Hochschild differential in $C^{n-p}(\mathcal{A})$ and $[\cdot, \cdot]$ on the right denote the Gerstenhaber brackets (in particular in the third line this amounts to the substitution of $\widehat{\rho}^2(X_i, X_j)$ into the Hochschild cochain

$$f_{p-2}(X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_p),$$

where we traditionally denote the missing arguments by the hat $\widehat{\ }$ above them). A straightforward computation then shows that $d^2 = 0$.

Remark 12. One can consider the maps $\widehat{\rho}^1, \widehat{\rho}^2$ as a homogeneous map

$$\Phi : \wedge^* \mathfrak{g} \to CH^{\cdot}(\mathcal{A}),$$

given by

$$\Phi(X_1 \wedge \cdots \wedge X_k) = \begin{cases} \widehat{\rho}^1(X_1), & k = 1, \\ \widehat{\rho}^2(X_1, X_2), & k = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Evidently we have Φ $\quad CHE^2(\mathfrak{g}, \mathcal{A})$ and $d_{CHE}\Phi = 0$. One can ask, what is cohomological meaning that one can attribute to the class of Φ? In particular, what one can say about the complex, if this class vanishes.

Consider now the natural projection i_* of $C^{\cdot}(\mathcal{A})$ to $C^{\cdot}(C^\infty(M))$ given by the inclusion of $C^\infty(M)$ into $\mathcal{A}$ and the inverse projection $\mathcal{A} \to C^\infty(M)$ (setting $\hbar = 0$). Since we have the condition of $\hbar$-linearity, this map is a homomorphism of chain complexes; clearly enough, it is epimorphic.Composing it with the linear maps from $\wedge^* \mathfrak{g}$ we obtain the following short exact sequence of cochain complexes:

$$0 \to CHE^{\cdot}(\mathfrak{g}, \mathcal{A}; C^\infty(M)) \to CHE^{\cdot}(\mathfrak{g}, \mathcal{A}) \to CHE^{\cdot}(\mathfrak{g}, C^\infty(M)) \to 0. \qquad (21)$$

Here $CHE^{\cdot}(\mathfrak{g}, C^\infty(M))$ is the (usual) Chevalley–Eilenberg cohomology complex of $\mathfrak{g}$ with coefficients in the Hochschild cohomology complex of $C^\infty(M)$ with standard differential and $CHE^{\cdot}(\mathfrak{g}, \mathcal{A}; C^\infty(M))$ is the kernel of the projection map.

Using the Lie algebra structure on $C^{\cdot}(\mathcal{A})$ (the Gerstenhaber bracket) we can introduce the Lie bracket on the complex $CHE^{\cdot}(\mathfrak{g},\mathcal{A})$: put

$$[\{f\},\{g\}](X_1,\dots,X_p)=\sum_{i=0}^{p}\sum_{\sigma\in Sh_{i,p-i}}(-1)^{\sigma(i,p,l)}$$
$$\times\,[f_i(X_{\sigma(1)},\dots,X_{\sigma(i)}),g_{p-i}(X_{\sigma(i+1)},\dots,X_{\sigma(p)})],\quad (22)$$

where $\sigma \quad Sh(i,p-i)$ is a $(i,p-i)$-shuffle, the sign $(-1)^{\sigma(i,p,l)}$ (l being the degree of g) is determined by the standard Koszul rules, and $[,]$ is the Gerstenhaber bracket (compare this formula with (20)). A simple calculation shows that this bracket verifies the standard identities, including the (shifted)Jacoby identity (shift is necessary since the degree of Gerstenhaber bracket is -1) so, we should shift the dimensions in $CHE^{\cdot}(\mathfrak{g},\mathcal{A})$. Clearly, the projection i_* commutes with the brackets, so the shifted subcomplex $CHE^{\cdot}(\mathfrak{g},\mathcal{A};C^{\infty}(M))[1]$ is also endowed with the natural differential graded Lie algebra structure.

Finally, let us consider the subspace $\widehat{CHE}^{\cdot}_1(\mathfrak{g},\mathcal{A})$ in $CHE^{\cdot}(\mathfrak{g},\mathcal{A};C^{\infty}(M))[1]$, spanned by the maps with values in the 1-dimensional Hochschild cocycles, i.e. in $\mathrm{Der}(\mathcal{A})$. Since $\mathrm{Der}(\mathcal{A})$ is a Lie subalgebra in the Hochschild complex $CH^{\cdot}(\mathcal{A})$, $\widehat{CHE}^{\cdot}_1(\mathfrak{g},\mathcal{A})$ is in effect a DG Lie subalgebra (with respect to the restrictions of the differential d_{CHE} and the bracket (22)). Then the formula

$$\Phi_1(X,Y)=\mathrm{ad}_{\widehat{\rho}^2(X,Y)}\quad Der_{\hbar}(\mathcal{A})$$

determines a cocycle in $\widehat{CHE}^{2}_1(\mathfrak{g},\mathcal{A})$. Now the following proposition is almost immediate from the definitions

Proposition 13. *One can change the map* $\widehat{\rho}^1:\mathfrak{g}\to\mathrm{Der}(\mathcal{A})$ *to a Lie algebra homomorphism* $\widehat{\rho}:\mathfrak{g}\to\mathrm{Der}(\mathcal{A})$ *so that* $\widehat{\rho}\equiv\widehat{\rho}^1 \mod \hbar$ *iff the* Φ_1 *is a "curvature" of some element* $\xi \quad \widehat{CHE}^{1}_1(\mathfrak{g},\mathcal{A})$*, i.e.*

$$\Phi_1=d_{CHE}\xi+\frac{1}{2}[\xi,\xi].$$

Proof. This is just a reinterpretation of the formula: $[\widehat{\rho}^1(X)+\xi(X),\widehat{\rho}^1(Y)+\xi(Y)]=\widehat{\rho}^1([X,Y])+\xi([X,Y])$. □

5. Conclusion: remarks and questions

In this paper we have described a large class of cohomology spaces and classes in them, corresponding in one or another way to the problem of finding the commutative family of elements in the quantized algebra of functions on a Poisson manifold. Although the list is by all means not complete, we hope it is big enough to make this problem look interesting and having wide intricate connections with many different constructions in Poisson Geometry and related branches of Mathematics. In particular, we hope that even though the original problem is still far from being completely solved (so far even an example of non-quantizable integrable system is lacking), some of the methods and approaches described in this paper can be used in other related research.

For instance, it is clear that all the classes described here are invariants of the Hamiltonian torus actions (more generally of Lie algebras actions by Poisson fields) on Poisson manifolds, so that is they do not coincide for two given examples, then the corresponding manifolds can not be equivariantly diffeomorphic. Thus the classes described in this paper can be used to discriminate non-isomorphic equivariant Poisson structures. It would be also very interesting to establish their relation with other existing approaches to similar questions, for instance with equivariant cohomology theories and other homotopy-theoretic constructions.

Another problem, related with the obstruction classes that we described here is to describe their relation with the geometry of the singular foliation, corresponding to the toric action. in fact, as we mentioned in the first section, property that we consider depend on the global properties of the trajectories of the integrable system. There exist many different geometric invariants of (singular) foliations in general and of foliations by Liouville tori in particular. For instance in the case of symplectic -dimensional manifolds, when all integrable systems consist of just two functions, there exist an elaborate system of complete invariants, due to Fomenko and Zieschang (see [13]). It would be interesting to describe some of the classes described in this paper in terms of these invariants.

Let us also mention the important potential role of Atiyah class, described in the section 3.1. Similar classes and their combinations are crucial for many problems. For instance, they appear in Calaque's and van den Bergh's study of Duflo isomorphism (see [14]). One can ask, if our classes have similar meaning, and also what are the properties of the Todd class and other functions of the Atiyah class presented here.

Apart from the questions, concerning the relation of the constructions we consider in this paper with other theories, there is a number of problems, concerned just with the constructions, described in this paper. First of all, let us observe that there are more than ten different constructions, mentioned here. They all are related with the same problem, so it is natural to think, that they are closely related to each other. In some cases we are able to establish such relations; for instance, we can show that the question of finding the Maurer–Cartan element in the differential Lie algebra $\bar{C}^*(\mathfrak{g}, C^*(C^\infty(M)[[\hbar]]))[1]$ (see Proposition 10, Section 4.1) is closely related to the Garay and van Straten classes (see Remark 11).

Besides this, the construction of Section 4.1 is clearly related to Proposition 13, although it is not evident, if two questions are equivalent, or one of them follows from the other. In general, one can suppose that the class, corresponding to $\widehat{\rho}^2$ (see Remark 11) and the class of Φ_1 from Proposition 13 should coincide, although the complexes in which they appear are not quite the same. All this is also closely related with what can be thought as the "relative formality": as we mentioned in Remark 11, the Lie algebra $C^*(\mathfrak{g}, C^*(C^\infty(M)[[\hbar]]))[1]$ is formal (at least for $M = \mathbb{R}^n$, where we can use an equivariant version of Kontsevich's map). Meanwhile the problem we consider here is related to the formality of $\bar{C}^*(\mathfrak{g}, C^*(C^\infty(M)[[\hbar]]))[1]$, which is the kernel of certain morphism of Lie algebras. Similar question appeared in the paper [2], where we considered the exact sequence of Lie algebras, given by the Hochschild complexes. At this moment we know no references, in which this question is treated (a close question was considered in [15]).

Finally, one more interesting problem, related with the classes we constructed, is *what form they take if the generic Poisson structure on M is replaced by some symplectic form?* In this case for example Poisson cohomology can be identified with the de Rham cohomology, so in many cases we shall obtain classes in the usual real cohomology theory of the manifold. It would be interesting to find explicit expressions for these classes as given by closed differential forms on M.

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Co-Toeplitz Quantization: A Simple Case

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Abstract. The author has introduced in a recent paper a new class of operators, called co-Toeplitz operators, with symbols in a co-algebra. This is the categorical dual to Toeplitz operators which have symbols in an algebra. The mapping from a symbol to its co-Toeplitz operator gives a quantization scheme, called co-Toeplitz quantization. A new, quite simple particular case of co-Toeplitz quantization is introduced in this note. Examples are given in order to show some of its properties.

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Keywords. Quantization, co-Toeplitz operators, co-algebras.

1. Introduction

In [2] I have defined co-Toeplitz operators in a dual way in terms of category theory to Toeplitz operators. The structures needed for this definition are a co-algebra $\mathcal{C}$ (see [1]) together with a sesqui-linear form $\langle \cdot, \cdot \rangle$ defined on it. We let $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ denote the co-multiplication of $\mathcal{C}$. Also, we suppose there is another co-algebra $\mathcal{P}$ which injects into $\mathcal{C}$ by a map $j : \mathcal{P} \to \mathcal{C}$ and that there is a projection $Q : \mathcal{C} \to \mathcal{P}$, that is, $Qj = id_{\mathcal{P}}$, the identity on $\mathcal{P}$. Then we define the *co-Toeplitz operator* $C_g : \mathcal{P} \to \mathcal{P}$ to be the linear operator defined by the composition

$$\mathcal{P} \xrightarrow{j} \mathcal{C} \xrightarrow{\Delta} \mathcal{C} \otimes \mathcal{C} \xrightarrow{Q \otimes id} \mathcal{P} \otimes \mathcal{C} \xrightarrow{\pi_g} \mathcal{P}. \tag{1}$$

The linear map $\pi_g : \mathcal{P} \otimes \mathcal{C} \to \mathcal{P}$, where $g \quad \mathcal{C}$ is called the *symbol* of the co-Toeplitz operator C_g, is defined in [2] by

$$\pi_g(\phi \otimes f) := \langle g, f \rangle \phi$$

for $\phi \quad \mathcal{P}$ and $f \quad \mathcal{C}$. (The map π_g is dual to α_g to be defined below.) The *anti-linear* map $g \mapsto C_g$ is called the *co-Toeplitz quantization* of $\mathcal{C}$. This in general does not involve measure theory. For more details see [2].

The definition (1) is the dual diagram to that for a *Toeplitz operator* which is defined for a symbol $g \quad \mathcal{A}$, an algebra, as the composition (right to left)

$$\mathcal{P} \xleftarrow{P} \mathcal{A} \xleftarrow{\mu} \mathcal{A} \otimes \mathcal{A} \xleftarrow{\iota \otimes id} \mathcal{P} \otimes \mathcal{A} \xleftarrow{\alpha_g} \mathcal{P}. \tag{2}$$

Here $\mathcal{P}$ is a sub-algebra of $\mathcal{A}$ with $\iota : \mathcal{P} \to \mathcal{A}$ being the inclusion map and $P : \mathcal{A} \to \mathcal{P}$ being a projection. Also, μ is the multiplication map of $\mathcal{A}$ and $\alpha_g(\phi) := \phi \otimes g$ for $\phi \quad \mathcal{P}$.

The definition (1) has a particular case: $\mathcal{P} = \mathcal{C}$ and $j = Q = id_{\mathcal{C}}$. Then

$$C_g = \pi_g \, \Delta : \mathcal{C} \to \mathcal{C}.$$

In this particular simple case, which is the new idea in this note, the only structures needed are a co-algebra and a sesqui-linear form on it. All the examples in this note fall within this case. This simple case does not have an interesting analogue for Toeplitz operators, since diagram (2) reduces to the right regular representation of g acting on $\mathcal{A}$ if we put $\mathcal{P} = \mathcal{A}$.

If the sesqui-linear form is positive definite, then C_g acts in a pre-Hilbert space and so may be considered as a densely defined operator acting in the Hilbert space completion of $\mathcal{C}$. Thus we can construct models for quantum physics, including creation and annihilation co-Toeplitz operators (see [2]).

All objects in this paper are vector spaces over the complex numbers, and all arrows are linear maps, except as noted.

2. Manin Quantum Plane

We define $\mathcal{C}$ to be the algebra generated by two elements a, c with the relation $ac = qca$ for some non-zero $q \quad \mathbb{C}$, the complex numbers. This is called the *Manin quantum plane*. The notation follows that used in [2]. We define the co-multiplication Δ to be the algebra morphism determined by

$$\Delta(a) = a \otimes a \quad \text{and} \quad \Delta(c) = c \otimes a.$$

This is well defined on $\mathcal{C}$, since $\Delta(ac - qca) = 0$ as the reader can verify. We note that $\mathcal{C}$ does not have a co-unit, though this has no great importance for our purposes. Clearly, $\mathcal{B} := \{a^i c^j \mid i, j \quad \mathbb{N}\}$, is a Hamel basis of $\mathcal{C}$, where $\mathbb{N}$ denotes the non-negative integers. Since C_g is anti-linear in the symbol $g \quad \mathcal{C}$, it suffices to calculate C_g for the basis elements $a^i c^j$. And since C_g is linear, it suffices to evaluate it on these basis elements. We proceed to do this. First, we see that

$$\Delta(a^k c^l) = \big(\Delta(a)\big)^k \big(\Delta(c)\big)^l = (a \otimes a)^k (c \otimes a)^l = a^k c^l \otimes a^{k+l}.$$

Then

$$C_{a^i c^j}(a^k c^l) = \pi_{a^i c^j} \, \Delta(a^k c^l) = \pi_{a^i c^j} \big(a^k c^l \otimes a^{k+l}\big) = \langle a^i c^j, a^{k+l} \rangle \, a^k c^l.$$

So $C_{a^i c^j}$ is diagonalized by the basis $\mathcal{B}$ with its eigenvalues determined by the sesqui-linear form, and therefore it is neither a creation nor an annihilation operator. Rather $C_{a^i c^j}$ is what is known as a *preservation operator*.

At this point in the calculation it becomes clear that the definition of the sesqui-linear form enters in a fundamental way, namely, different choices for it give different co-Toeplitz quantizations. One simple choice is to choose it so that the basis $\mathcal{B}$ is orthogonal. So we put

$$\langle a^i c^j, a^k c^l \rangle := \delta_{i,k}\, \delta_{j,l}\, w(i,j).$$

Here $\delta_{\ ,s}$ denotes the Kronecker delta. We also take $w(i,j) > 0$ for all $i, j \quad \mathbb{N}$ so that this is a positive definite inner product and $\mathcal{C}$ is a pre-Hilbert space. With this inner product we see that

$$C_{a^i c^j}(a^k c^l) = \langle a^i c^j, a^{k+l} \rangle\, a^k c^l = \delta_{i,k+l}\, \delta_{j,0}\, w(i,j)\, a^k c^l.$$

So $C_{a^i c^j} = 0$ if $j > 0$. Continuing with the case $j = 0$ we see that

$$C_{a^i}(a^k c^l) = \delta_{i,k+l}\, w(i,0)\, a^k c^l.$$

So C_{a^i} is diagonalized by the basis $\mathcal{B}$ with 0 and $w(i,0)$ being its eigenvalues. We define the degree of the monomial $a^k c^l$ by $\deg\, a^k c^l := k + l$. Then C_{a^i} is zero on monomials with degree $\neq i$ and is a non-zero multiple of the identity on the finite dimensional vector space spanned by the monomials of degree i.

Another choice for the sesqui-linear form is

$$\langle a^i c^j, a^k c^l \rangle := \delta_{i-j,k-l}\, \mu(i,j,k,l),$$

where $\mu : \mathbb{N}^4 \to (0, \infty)$ is a positive weight function. With this choice of sesqui-linear form we have

$$C_{a^i c^j}(a^k c^l) = \langle a^i c^j, a^{k+l} \rangle\, a^k c^l = \delta_{i-j,k+l}\, \mu(i,j,k+l,0)\, a^k c^l.$$

Thus the eigenvalues of $C_{a^i c^j}$ are 0 and $\mu(i,j,i-j,0)$. Moreover, $C_{a^i c^j}$ is zero except on the set of monomials of degree $i - j$. So $i < j$ implies that $C_{a^i c^j} = 0$.

3. Divided Power Co-algebra

This is based on Example 2.4.8 in [1]. We let $\mathcal{C}$ be the vector space with basis $\{x_i \mid i \quad \mathbb{N}\}$. The co-multiplication Δ is the linear map determined by

$$\Delta(x_n) := \sum_{i+j=n} x_i \otimes x_j. \tag{3}$$

The degree of each basis element is defined by $\deg x_n := n$. We also define a sesqui-linear form by

$$\langle x_i, x_j \rangle := w(i)\, \delta_{i,j}, \tag{4}$$

where $w : \mathbb{N} \to (0, \infty)$ is a strictly positive weight function. So this is an inner product making $\mathcal{C}$ into a pre-Hilbert space. Again, it suffices to compute the co-Toeplitz operators T_g for g in the basis. So we compute as follows:

$$C_{x_k}(x_n) = \pi_{x_k}\Delta(x_n) = \pi_{x_k}\Bigg(\sum_{i+j=n} x_i \otimes x_j\Bigg)$$
$$= \sum_{i+j=n} \langle x_k, x_j\rangle\, x_i = \sum_{i+j=n} \delta_{k,j} w(k)\, x_i.$$

Now if $k > n$ we have $\delta_{k,j} = 0$ for all the terms in the last sum, since $j \le n$. So $C_{x_k}(x_n) = 0$ if $k > n$. For the opposite case $0 \le k \le n$ we have

$$C_{x_k}(x_n) = \sum_{i+j=n} \delta_{k,j} w(k)\, x_i = w(k)\, x_{n-k}.$$

If we define $x_i := 0$ and $w(i) := 0$ for all integers $i < 0$, then we can write this result as one formula for all $k, n \;\; \mathbb{N}$:

$$C_{x_k}(x_n) = w(k)\, x_{n-k}.$$

So for $k > 0$ we have that C_{x_k} decreases degree by k and so is an annihilation operator. On the other hand C_{x_0} is a preservation operator.

4. Negative Degrees

Here we give a modification of the previous example that includes negative degrees. We let $M \ge 1$ be an integer and define $\mathcal{C}$ to be the complex vector space with basis $\{x_i\}$ for integers $i \;\; [-M, M]$. So $\dim \mathcal{C} = 2M + 1$.

We define $\deg x_i := i$ for $i \;\; [-M, M]$. For convenience we also define $x_i := 0$ for all integers i with $|i| > M$. We use the same formulas as in the previous example, but with new interpretations. So, the co-multiplication Δ is defined by (3), but now for integers $|n| \le M$. With our definitions only finitely many terms in the (now) infinite sum (3) are non-zero. We also define a sesqui-linear form by (4) but now for integers $i, j \;\; [-M, M]$. Again, for convenience we put $w(i) := 0$ for $|i| > M$. The same calculation as in the previous example gives

$$C_{x_k}(x_n) = w(k)\, x_{n-k}$$

but now for all integers $k, n \;\; [-M, M]$. There are three cases:

1. C_{x_k} increases degree by $|k|$ if $k < 0$ and is a creation operator.
2. C_{x_k} decreases degree by k if $k > 0$ and is an annihilation operator.
3. C_{x_k} preserves degree if $k = 0$ and is a preservation operator.

So we get the three types of operators relevant to physics by using the basis elements with positive, negative and zero degrees. We also can define a $*$-*operation* ($\equiv$ *conjugation*) on $\mathcal{C}$ by putting $x_i^* := x_{-i}$. This definition is motivated by the theory of complex variables. Using this as motivation, for each $i > 0$ we then define the elements x_i to be *holomorphic* and the elements x_i^* to be *anti-holomorphic*.

(The element x_0 could be defined as being both holomorphic and anti-holomorphic, if one wished. But I opt not to do that.) Then the anti-holomorphic elements are symbols of creation operators while the holomorphic elements are symbols of annihilation operators.

5. Matrix Co-algebra

This example comes from Example 2.4.1 in [1]. Let $\mathcal{C}$ be the vector space with basis $\{E_{i,j} \mid 1 \leq i, j \leq n\}$, where $n \geq 1$ is an integer. So, $\dim \mathcal{C} = n^2$. Of course, the motivation is that $E_{i,j}$ is analogous to the $n \times n$ matrix with all entries 0, except for row i and column j which has the entry 1. Let the co-multiplication be determined by

$$\Delta(E_{i,j}) := \sum_{k=1}^{n} E_{i,k} \otimes E_{k,j}.$$

(As a curious parenthetical remark, let us note that this vector space has a natural algebra structure motivated by matrix multiplication. However, this does not combine with this co-multiplication to give us a bi-algebra; see [1].) We define an inner product on $\mathcal{C}$ by making the basis $\{E_{i,j}\}$ orthonormal. Then we calculate

$$\begin{aligned} C_{E_{r,s}}(E_{i,j}) = \pi_{E_{r,s}}\, \Delta(E_{i,j}) &= \pi_{E_{r,s}} \left(\sum_{k=1}^{n} E_{i,k} \otimes E_{k,j} \right) \\ &= \sum_{k=1}^{n} \langle E_{\ ,s}, E_{k,j} \rangle E_{i,k} = \sum_{k=1}^{n} \delta_{\ ,k} \delta_{s,j} E_{i,k} = \delta_{s,j} E_{i,\ }\ . \end{aligned}$$

So, $C_{E_{r,s}}(E_{i,j})$ is either zero or another basis element.

Another sesqui-linear form is given by $\langle E_{i,j}, E_{\ ,s} \rangle := w(i+s)\, \delta_{i-j,\ -s}$ with a weight function $w : \mathbb{N} \to (0, \infty)$. Then

$$\begin{aligned} C_{E_{r,s}}(E_{i,j}) = \pi_{E_{r,s}}\, \Delta(E_{i,j}) &= \pi_{E_{r,s}} \left(\sum_{k=1}^{n} E_{i,k} \otimes E_{k,j} \right) \\ &= \sum_{k=1}^{n} \langle E_{\ ,s}, E_{k,j} \rangle E_{i,k} = \sum_{k=1}^{n} w(r+j) \delta_{\ -s,k-j}\, E_{i,k} = w(r+j) E_{i,j+\ -s}, \end{aligned}$$

where we put $E_{i,j} = 0$ if $j \leq 0$ or $j > n$. If we define $\deg E_{i,j} := i + j$, then we see that $C_{E_{r,s}}$ changes degree by $r - s$. So, $C_{E_{r,s}}$ is a creation operator if $r > s$, it is an annihilation operator if $r < s$ and finally it is a preservation operator if $r = s$.

Using the adjoint operation of matrices as motivation, we also define a $*$-operation by $E_{i,j}^* := E_{j,i}$. We also say that $E_{i,j}$ is *holomorphic* if $i < j$ ('upper triangular') and is *anti-holomorphic* if $i > j$ ('lower triangular'). As previously, the anti-holomorphic $E_{i,j}$ are the symbols of (degree increasing) creation operators and, on the other hand, the holomorphic $E_{i,j}$ are the symbols of (degree decreasing) annihilation operators. Also the 'diagonal' elements $E_{i,i}$, which are self-adjoint (or real) with respect to the $*$-operation, are symbols of (degree preserving) preservation operators.

6. Concluding Remarks

The quantization of co-algebras is a new field of research with co-Toeplitz quantization being the first theory that achieves this. It is remarkable that any sesqui-linear form defined on a co-algebra $\mathcal{C}$ is sufficient extra structure to give us a co-Toeplitz quantization of $\mathcal{C}$. It is noteworthy that in some of these examples a $*$-operation can be defined thereby giving holomorphic and anti-holomorphic elements, which are symbols whose co-Toeplitz operators are annihilation and creation operators, respectively.

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Part III

Quantum groups and non-commutative geometry

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On the quantum flag manifold $SU_q(3)/\mathbb{T}^2$

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Abstract. The structure of the C^*-algebra of functions on the quantum flag manifold $SU_q(3)/\mathbb{T}^2$ is investigated. Building on the representation theory of $C\big(SU_q(3)\big)$, we analyze irreducible representations and the primitive ideal space of $C\big(SU_q(3)/\mathbb{T}^2\big)$, with a view towards unearthing the "quantum sphere bundle" $\mathbb{C}P^1_q \to SU_q(3)/\mathbb{T}^2 \to \mathbb{C}P^2_q$.

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1. Introduction

The theory of principal and associated fibre bundles lies at the heart of geometry and underpins important applications to physics. Due to combined effort of many researchers, see e.g. [2, 3, 6], this theory has been successfully incorporated into noncommutative geometry. In the noncommutative setting, spaces are replaced by (noncommutative) algebras of functions, typically C^*-algebras or their dense $*$-subalgebras, and quantum groups (or Hopf algebras) play the role of structure groups. By contrast, precious little is known about noncommutative analogs of more general fibre bundles, in which the fibre does not correspond to a group.

This short note is intended as a first step towards a case study of noncommutative sphere bundles. More specifically, the classical flag manifold $SU(3)/\mathbb{T}^2$ has a natural structure of the sphere bundle

$$\mathbb{C}P^1 \to SU(3)/\mathbb{T}^2 \to \mathbb{C}P^2.$$

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We intend to analyze the structure of the quantum analog of this flag manifold, corresponding to the C^*-algebra $C\big(SU_q(3)/\mathbb{T}^2\big)$ playing the role of the total space. Here $SU_q(3)$ denotes the Woronowicz quantum $SU(3)$ group, and $C\big(SU_q(3)/\mathbb{T}^2\big)$ itself is the C^*-algebra of fixed points for the action of $\mathbb{T}^2$ on $C\big(SU_q(3)\big)$ coming from its maximal torus.

The quantum flag manifold $SU_q(3)/\mathbb{T}^2$ is just one of the large family of (generalised) quantum flag manifolds, whose structure has been studied and described in full generality in [12] and [9]. However, in order to be able to understand $SU_q(3)/\mathbb{T}^2$ as the total space of a quantum sphere bundle from the analytic point of view, it is necessary to have a detailed and explicit information about the internal structure of the C^*-algebra $C\big(SU_q(3)/\mathbb{T}^2\big)$ readily accessible. This is the main aim of the present note. In particular, we carefully describe the primitive ideal space of this AF-algebra, building on the explicit description of irreducible representations of $C\big(SU_q(3)\big)$, calculated originally by K. Brągiel in his PhD dissertation [1].

In the final section of this note we show how to construct a faithful conditional expectation from $C\big(SU_q(3)/\mathbb{T}^2\big)$ onto its subalgebra $C\big(\mathbb{C}P_q^2\big)$, using integration over the quantum group $U_q(2)$ (realised as a quantum subgroup of $SU_q(3)$). More detailed, algebraic description of the noncommutative sphere bundle

$$\mathbb{C}P_q^1 \to SU(3)/\mathbb{T}^2 \to \mathbb{C}P_q^2$$

and its K-theory is deferred to the forthcoming paper [4].

2. The quantum flag manifold

2.1. The algebra of functions on the quantum SU(3) group

For q $(0,1)$, the C^*-algebra $C\big(SU_q(3)\big)$ of 'continuous functions' on the quantum $SU(3)$ group is defined by Woronowicz [15, 16] as the universal C^*-algebra generated by elements $\{u_{ij} : i,j = 1,2,3\}$ such that the matrix $\mathbf{u} = (u_{ij})_{i,j=1}$ is unitary and

$$\sum_{i_1=1}\sum_{i_2=1}\sum_{i\ \ =1} E_{i_1 i_2 i}\; u_{j_1 i_1} u_{j_2 i_2} u_{j\ \ i}\ \ = E_{j_1 j_2 j}\ , \qquad \forall (j_1, j_2, j\ \ )\quad \{1,2,3\},$$

where

$$E_{i_1 i_2 i}\ \ = \begin{cases} (-q)^{I(i_1,i_2,i\ \ )} & \text{if } i\ \ \neq i_s \text{ for } r \neq s, \\ 0 & \text{otherwise,} \end{cases}$$

and $I(i_1, i_2, i\ \ )$ denotes the number of inversed pairs in the sequence $i_1, i_2, i\ \ $. As pointed out by Brągiel [1], $\{u_{ij}\}$ are coordinate functions of a quantum matrix

[5, 10, 11]. That is, the following relations are also satisfied

$$u_{ij}u_{ik} = qu_{ik}u_{ij}, \qquad j < k, \tag{1a}$$
$$u_{ji}u_{ki} = qu_{ki}u_{ji}, \qquad j < k, \tag{1b}$$
$$u_{ij}u_{km} = u_{km}u_{ij}, \qquad i < k,\ j > m, \tag{1c}$$
$$u_{ij}u_{km} - u_{km}u_{ij} = (q - q^{-1})u_{im}u_{kj}, \qquad i < k,\ j < m, \tag{1d}$$

with $i, j, k, m \quad \{1, 2, 3\}$. The comultiplication

$$\Delta : C\big(SU_q(3)\big) \longrightarrow C\big(SU_q(3)\big) \otimes C\big(SU_q(3)\big)$$

is a unital C^*-algebra homomorphism such that

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}.$$

We denote by $\mathcal{O}\big(SU_q(3)\big)$ the $*$-subalgebra of $C\big(SU_q(3)\big)$ generated by the u_{ij}, $i, j = 1, 2, 3$. Thus $\mathcal{O}\big(SU_q(3)\big)$, the polynomial algebra of $SU_q(3)$, is a dense $*$-subalgebra of $C\big(SU_q(3)\big)$.

In [1], Brągiel described explicitly all irreducible representations of the algebra $C\big(SU_q(3)\big)$. There are six families of these representations, each indexed by elements (ϕ, ψ) of the 2-torus. We denote them by $\pi_0^{\phi,\psi}$, $\pi_{11}^{\phi,\psi}$, $\pi_{12}^{\phi,\psi}$, $\pi_{21}^{\phi,\psi}$, $\pi_{22}^{\phi,\psi}$ and $\pi^{\phi,\psi}$. Each of the representations $\pi_*^{\phi,\psi}$ acts on the Hilbert space $\mathcal{H}_*$, where

$$\mathcal{H}_0 = \mathbb{C},\ \mathcal{H}_{11} = \mathcal{H}_{12} = \ell^2(\mathbb{N}),\ \mathcal{H}_{21} = \mathcal{H}_{22} = \ell^2\big(\mathbb{N}^2\big) \text{ and } \mathcal{H}\ \ = \ell^2\big(\mathbb{N}\ \ \big).$$

Each of the $\pi_*^{\phi,\psi}$ contains compact operators of $\mathcal{H}_*$ in its image [1], and thus $C\big(SU_q(3)\big)$ is a type I algebra. The kernels of these irreducible representations are primitive ideals of $C\big(SU_q(3)\big)$ with the following generators:

$$\ker\big(\pi^{\phi,\psi}\big) = \big\langle \overline{\phi}u_{\ 1} - |u_{\ 1}|,\ \overline{\psi}u_{1\ } - |u_{1\ }|\big\rangle, \tag{2a}$$
$$\ker\big(\pi_{21}^{\phi,\psi}\big) = \big\langle u_{\ 1},\ \overline{\phi}u_{21} - |u_{21}|,\ \overline{\psi}u_{1\ } - |u_{1\ }|\big\rangle, \tag{2b}$$
$$\ker\big(\pi_{22}^{\phi,\psi}\big) = \big\langle u_{1\ },\ \overline{\phi}u_{\ 1} - |u_{\ 1}|,\ \overline{\psi}u_{12} - |u_{12}|\big\rangle, \tag{2c}$$
$$\ker\big(\pi_{11}^{\phi,\psi}\big) = \big\langle u_{1\ },\ u_{\ 1},\ u_{2\ },\ \overline{\phi}u_{12} - |u_{12}|,\ \overline{\psi}u_{21} - |u_{21}|\big\rangle, \tag{2d}$$
$$\ker\big(\pi_{12}^{\phi,\psi}\big) = \big\langle u_{1\ },\ u_{\ 1},\ u_{12},\ \phi\psi u_{\ 2} - |u_{\ 2}|,\ \overline{\psi}u_{2\ } - |u_{2\ }|\big\rangle, \tag{2e}$$
$$\ker\big(\pi_0^{\phi,\psi}\big) = \big\langle u_{1\ },\ u_{\ 1},\ u_{12},\ u_{2\ },\ \overline{\phi}u_{11} - 1,\ \overline{\psi}u_{22} - 1\big\rangle. \tag{2f}$$

2.2. The gauge action and its fixed point algebra

The family of 1-dimensional irreducible representations $\pi_0^{\phi,\psi}$ of $C\big(SU_q(3)\big)$ produces a surjective morphism of compact quantum groups

$$\hat{\pi}_0 : C\big(SU_q(3)\big) \longrightarrow C\big(\mathbb{T}^2\big)$$

(the diagonal imbedding of $\mathbb{T}^2$ into $SU_q(3)$), which gives rise to a gauge coaction of coordinate algebras

$$\hat{\mu} : \mathcal{O}\big(SU_q(3)\big) \to \mathcal{O}\big(SU_q(3)\big) \otimes \mathcal{O}\big(\mathbb{T}^2\big), \qquad \hat{\mu} = (\mathrm{id} \otimes \hat{\pi}_0) \circ \Delta_{SU_q(\ )}.$$

Explicitly, on the polynomial algebra $\mathcal{O}(SU_q(3))$, $\hat{\pi}_0$ is a Hopf $*$-algebra epimorphism,

$$\hat{\pi}_0 : \mathcal{O}\big(SU_q(3)\big) \longrightarrow \mathcal{O}(\mathbb{T}^2), \qquad \mathbf{u} \mapsto \begin{pmatrix} U_1 & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & U_1^* U_2^* \end{pmatrix},$$

where U_1, U_2 are unitary, group-like generators of the Hopf algebra $\mathcal{O}(\mathbb{T}^2)$ of polynomials on $\mathbb{T}^2$ (the algebra of Laurent polynomials in two indeterminates). Hence the coaction comes out as

$$\hat{\mu} : \mathcal{O}\big(SU_q(3)\big) \to \mathcal{O}\big(SU_q(3)\big) \otimes \mathcal{O}\big(\mathbb{T}^2\big), \quad u_{ij} \mapsto \begin{cases} u_{ij} \otimes U_j & \text{if } j = 1, 2, \\ u_{ij} \otimes (U_1 U_2)^{-1} & \text{if } j = 3. \end{cases}$$

Equivalently, $\mu : \mathbb{T}^2 \longrightarrow \mathrm{Aut}\,\big(C\big(SU_q(3)\big)\big)$ is given by

$$z \longmapsto \mu_z, \qquad \mu_z(u_{ij}) = \begin{cases} z_j u_{ij} & \text{if } j = 1, 2, \\ (z_1 z_2)^{-1} u_{ij} & \text{if } j = 3. \end{cases}$$

Here $z = (z_1, z_2) \quad \mathbb{T}^2$ and each z_i is a complex number of modulus 1. Let $C(SU_q(3)/\mathbb{T}^2)$ be the fixed point algebra of this gauge action, and let

$$\mathcal{O}\big(SU_q(3)/\mathbb{T}^2\big) = \mathcal{O}\big(SU_q(3)\big) \cap C\big(SU_q(3)/\mathbb{T}^2\big)$$

be its polynomial $*$-subalgebra, i.e. the subalgebra of coinvariants of $\hat{\mu}$,

$$\mathcal{O}\big(SU_q(3)/\mathbb{T}^2\big) = \mathcal{O}\big(SU_q(3)\big)^{\mathrm{co}\,\mathcal{O}(\mathbb{T}^2)} = \big\{f \quad \mathcal{O}\big(SU_q(3)\big) \,:\, \hat{\mu}(f) = f \otimes 1\big\}.$$

Integration with respect to the Haar measure over $\mathbb{T}^2$ gives rise to a faithful conditional expectation $\Phi : C\big(SU_q(3)\big) \to C\big(SU_q(3)/\mathbb{T}^2\big)$, namely

$$\Phi(x) = \int_{z \in \mathbb{T}^2} \mu_z(x) dz.$$

If w is a monomial in $\{u_{ij}\}$ then $\Phi(w)$ is either 0 or w. Thus we have

$$\Phi\big(\mathcal{O}\big(SU_q(3)\big)\big) = \mathcal{O}\big(SU_q(3)/\mathbb{T}^2\big),$$

and whence $\mathcal{O}\big(SU_q(3)/\mathbb{T}^2\big)$ is a dense $*$-subalgebra of $C\big(SU_q(3)/\mathbb{T}^2\big)$.

There is a third equivalent way of understanding the gauge action, which is particularly useful in determining the freeness of the action (alas we will not employ this point of view in this note): $\mathcal{O}(SU_q(3))$ is a $\mathbb{Z}^2$-graded algebra with the degrees of the generators given by

$$\deg(u_{i1}) = (1, 0), \ \deg(u_{i2}) = (0, 1), \ \deg(u_{i\ }) = (-1, -1), \quad i = 1, 2, 3.$$

From this point of view, $\mathcal{O}\big(SU_q(3)/\mathbb{T}^2\big)$ is the $(0, 0)$-degree part of $\mathcal{O}\big(SU_q(3)\big)$.

In what follows, we denote

$$w_{ijk} = u_{i1} u_{j2} u_{k\ }, \qquad i, j, k = 1, 2, 3. \tag{3}$$

Clearly, elements w_{ijk} are contained in the polynomial algebra $\mathcal{O}\big(SU_q(3)/\mathbb{T}^2\big)$.

Let $\rho_*^{\phi,\psi}$ be the restriction to $C\big(SU_q(3)/\mathbb{T}^2\big)$ of the representation $\pi_*^{\phi,\psi}$ of $C\big(SU_q(3)\big)$.

Lemma 1. *For each* $(\phi,\psi)\quad \mathbb{T}^2$*, the representation* $\rho_*^{\phi,\psi}$ *is unitarily equivalent to* $\rho_*^{1,1}$*.*

Proof. It follows immediately from formulae (2a)–(2f) that the gauge action μ on the primitive ideal space is transitive on each of the six families. Since $C\big(SU_q(3)\big)$ is of type I, irreducible representations with identical kernels are unitarily equivalent. Thus, for each (ϕ,ψ) there exist (z_1,z_2) such that $\pi_*^{\phi,\psi}$ is unitarily equivalent to $\pi_*^{1,1}\circ\mu_{z_1,z_2}$. But $\rho_*^{1,1}\circ\mu_{z_1,z_2}=\rho_*^{1,1}$, and so $\rho_*^{\phi,\psi}$ is unitarily equivalent to $\rho_*^{1,1}$. $\square$

In what follows we use the simplified notation $\rho_*=\rho_*^{1,1}$.

Lemma 2. *The image of* ρ_* *contains all the compact operators* $\mathcal{K}(\mathcal{H}_*)$ *on its space* $\mathcal{H}_*$*, and thus each* ρ_* *is irreducible.*

Proof. Representation ρ_0 is 1-dimensional and there is nothing to prove in this case.

Considering ρ_{12}, given by formulae (14) of [1], we have

$$\rho_{12}(w_{1\ \ 2})|N\rangle=-q^{2N+1}|N\rangle.$$

Thus the image of ρ_{12} contains one-dimensional projections corresponding to the basis $\{|N\rangle : N\quad \mathbb{N}\}$ of $\mathcal{H}_{12}$. Since

$$\rho_{12}(w_{1\quad})|N\rangle=\text{scalar}|N+1\rangle$$

(in the course of the proof of this lemma we denote by 'scalar' a non-zero constant which may depend on N,M,L), it follows that the image of ρ_{12} contains all the compact operators on $\mathcal{H}_{12}$. In the case of ρ_{11} the same argument works, since $\rho_{11}(w_{21\ \ })=\rho_{12}(w_{1\ \ 2})$ and $\rho_{11}(w_{22\ \ })|N\rangle=\text{scalar}|N+1\rangle$.

By formulae (12) of [1], $\rho_{22}(w_{\ \ 12})|N,M\rangle=q^{2(N+M+1)}|N,M\rangle$ and

$$\rho_{22}(w_{1\ \ 2})|N,M\rangle=-q^{2M+1}(1-q^{2(N+1)})|N,M\rangle.$$

It follows that the image of ρ_{22} contains all one-dimensional projections corresponding to the basis $\{|N,M\rangle : N,M\quad \mathbb{N}\}$ of $\mathcal{H}_{22}$. We also find that

$$\rho_{22}(w_{112})|N,M\rangle=\text{scalar}|N-1,M\rangle\quad\text{and}\quad\rho_{22}(w_{212})|N,M\rangle=\text{scalar}|N,M-1\rangle,$$

and it follows that the image of ρ_{22} contains all the compact operators on $\mathcal{H}_{22}$. The argument for ρ_{21} is similar and based on the identities:

$$\rho_{21}(w_{2\ \ 1})=\rho_{22}(w_{\ \ 12}),\qquad\qquad \rho_{21}(w_{1\ \ 2})=\rho_{22}(w_{1\ \ 2}),$$
$$\rho_{21}(w_{1\ \ 1})|N,M\rangle=\text{scalar}|N-1,M\rangle,\quad \rho_{21}(w_{211})|N,M\rangle=\text{scalar}|N,M-1\rangle.$$

Finally, considering $\rho_{\ \ }$, given by formulae (10) of [1], we have

$$\rho_{\ \ }\big(|w_{\ \ 11}|^2\big)|N,M,L\rangle=q^{2(\ \ N+M+L+\ \ )}\big(1-q^{2M}\big)|N,M,L\rangle, \tag{4a}$$
$$\rho_{\ \ }(w_{111})|N,M,L\rangle=\text{scalar}|N-1,M-1,L\rangle, \tag{4b}$$
$$\rho_{\ \ }(w_{211})|N,M,L\rangle=\text{scalar}|N,M-1,L-1\rangle, \tag{4c}$$
$$\rho_{\ \ }(w_{\ \ 11})|N,M,L\rangle=\text{scalar}|N,M-1,L\rangle. \tag{4d}$$

By (4a), the operator $\rho\ \left(|w\ _{11}|^2\right)$ is compact and its spectral subspace corresponding to the maximal eigenvalue is spanned by vectors $|0, M, 0\rangle$ for which M is a positive integer such that $q^{2M}\left(1 - q^{2M}\right)$ is maximal. This space is either one or two-dimensional. In the former case, the image of ρ contains the one-dimensional projection onto $|0, M_0, 0\rangle$, and formulae (4b)–(4d) imply that it contains all the compact operators on $\mathcal{H}$. In the latter case, the image of ρ contains the two-dimensional projection Q onto the span of $|0, M_0, 0\rangle$ and $|0, M_0 + 1, 0\rangle$. Then $Q\rho\ (w\ _{11})Q$ is a rank one operator, and just as above it follows from formulae (4b)–(4d) that the image of ρ contains all the compact operators on $\mathcal{H}$. □

We define $J_* = \rho_*^{-1}(\mathcal{K}(\mathcal{H}_*))$, closed ideals of $C\left(SU_q(3)/\mathbb{T}^2\right)$.

Lemma 3. *The following properties hold:*

(i) *Representation ρ of $C\left(SU_q(3)/\mathbb{T}^2\right)$ is faithful.*
(ii) $J\ = \ker(\rho_{21}) \cap \ker(\rho_{22})$.
(iii) $J_{21} = J_{22} = \ker(\rho_{21}) + \ker(\rho_{22}) = \ker(\rho_{11}) \cap \ker(\rho_{12})$.
(iv) $J_{11} = J_{12} = \ker(\rho_{11}) + \ker(\rho_{12}) = \ker(\rho_0)$.

Proof. We have

$$\ker(\rho_*) = C\left(SU_q(3)/\mathbb{T}^2\right) \cap \bigcap_{\phi,\psi} \ker\left(\pi_*^{\phi,\psi}\right).$$

Using formulae (2a)–(2f) we see that $\cap_{\phi,\psi} \ker\left(\pi_*^{\phi,\psi}\right)$ are ideals of $C\left(SU_q(3)\right)$ with the following sets of generators:

$$\bigcap_{\phi,\psi} \ker\left(\pi^{\phi,\psi}\right) = \langle 0\rangle, \tag{5a}$$

$$\bigcap_{\phi,\psi} \ker\left(\pi_{21}^{\phi,\psi}\right) = \langle u\ _1\rangle, \qquad \bigcap_{\phi,\psi} \ker\left(\pi_{22}^{\phi,\psi}\right) = \langle u_1\ \rangle, \tag{5b}$$

$$\bigcap_{\phi,\psi} \ker\left(\pi_{11}^{\phi,\psi}\right) = \langle u_1\ ,\ u\ _1,\ u_2\ \rangle, \tag{5c}$$

$$\bigcap_{\phi,\psi} \ker\left(\pi_{12}^{\phi,\psi}\right) = \langle u_1\ ,\ u\ _1,\ u_{12}\rangle, \tag{5d}$$

On the other hand, $J_* = \ker\left(\rho_*\right) \cap \langle x_*\rangle$, where $\langle x_*\rangle$ is the ideal of $C\left(SU_q(3)\right)$ generated by x_* such that $\pi_*^{\phi,\psi}(x_*)$ is a non-zero element of $\mathcal{K}(\mathcal{H}_*)$ for all ϕ, ψ. For example, we can take $x\ = u\ _1 u_1\ $, $x_{21} = u_1\ $, $x_{22} = u\ _1$, $x_{11} = u_{12}$ and $x_{12} = u_2\ $.

Now (i) follows from formula (5a). Claim (ii) follows from (5b) and the identity $\langle u\ _1\rangle \cap \langle u_1\ \rangle = \langle u\ _1 u_1\ \rangle$. The latter follows from the fact that both $u_1\ $ and $u\ _1$ either commute or q-commute with every generator of $C\left(SU_q(3)\right)$.

The identity $J_{21} = J_{22} = \ker(\rho_{21}) + \ker(\rho_{22})$ follows from (5b). For the remaining part of claim (iii), it suffices to show that

$$\langle u_1\ ,\ u\ _1\rangle = \langle u_1\ ,\ u\ _1,\ u_{12}\rangle \cap \langle u_1\ ,\ u\ _1,\ u_2\ \rangle.$$

To this end, we first note that modulo the ideal $\langle u_{1}\ ,\ u_{\ 1}\rangle$ both u_{12} and $u_{2}\ $ either commute or q-commute with every generator of $C\big(SU_q(3)\big)$. Thus

$$\langle u_{1}\ ,\ u_{\ 1},\ u_{12}\rangle \cap \langle u_{1}\ ,\ u_{\ 1},\ u_{2}\ \rangle = \langle u_{1}\ ,\ u_{\ 1},\ u_{12}u_{2}\ \rangle,$$

and it suffices to verify that $u_{12}u_{2}\quad \langle u_{1}\ ,\ u_{\ 1}\rangle$. By formulae (11) of [1], we have $\pi_{21}^{\phi,\psi}(u_{12}u_{2}\ )\quad \mathcal{K}(\mathcal{H}_{21})$ for all ϕ, ψ, and the claim follows.

The identity $J_{11} = J_{12} = \ker(\rho_{11}) + \ker(\rho_{12})$ follows from (5c) and (5d). For the remaining part of claim (iv), we must prove that

$$\ker(\rho_0) = C\big(SU_q(3)/\mathbb{T}^2\big) \cap \langle u_{1}\ ,\ u_{\ 1},\ u_{2}\ ,\ u_{12}\rangle.$$

However, as shown in [1], $\oplus_{\phi,\psi}\pi_0^{\phi,\psi}$ is faithful on the quotient of $C\big(SU_q(3)\big)$ by $\langle u_{1}\ ,\ u_{\ 1},\ u_{2}\ ,\ u_{12}\rangle$, and the claim follows. □

In the following corollary we summarise properties of the algebra of continuous functions on the quantum flag manifold $SU_q(3)/\mathbb{T}^2$.

Corollary 4. *The C^*-algebra $C\big(SU_q(3)/\mathbb{T}^2\big)$ has the following properties:*

1. *It has a composition series with factors: $\mathcal{K}$, $\mathcal{K}\oplus\mathcal{K}$, $\mathcal{K}\oplus\mathcal{K}$, $\mathbb{C}$.*
2. *It is AF and of type I.*
3. *Its K-groups are $K_0 \cong \mathbb{Z}^6$ and $K_1 = 0$.*
4. *$\{\rho_*\}$ is a complete set of representatives (up to unitary equivalence) of its irreducible representations.*
5. *Each irreducible representation of $C\big(SU_q(3)/\mathbb{T}^2\big)$ extends to an irreducible representation of $C\big(SU_q(3)\big)$ acting on the same Hilbert space.*
6. *Its primitive ideal space consists of six elements $\{\ker(\rho_*)\}$, with topology determined by the following closure operation.*
   (a) *The point* $\ker(\rho\ \ )$ *is dense in the entire space.*
   (b) *The closures of* $\ker(\rho_{21})$ *and* $\ker(\rho_{22})$, *respectively, consist of the union of itself and* $\{\ker(\rho_0), \ker(\rho_{11}), \ker(\rho_{12})\}$.
   (c) *The closures of* $\ker(\rho_{11})$ *and* $\ker(\rho_{12})$, *respectively, consist of the union of itself and* $\ker(\rho_0)$.
   (d) *The point* $\ker(\rho_0)$ *is closed.*

3. Towards a noncommutative sphere bundle

The classical flag manifold $SU(3)/\mathbb{T}^2$ has the structure of a fibre bundle with the base space $\mathbb{C}P^2$ and the fibre $\mathbb{C}P^1 \cong S^2$. Therefore, it is natural to expect that the quantum flag manifold $SU_q(3)/\mathbb{T}^2$ should have an analogous structure of a noncommutative 'fibre bundle'

$$\mathbb{C}P_q^1 \longrightarrow SU_q(3)/\mathbb{T}^2 \longrightarrow \mathbb{C}P_q^2. \tag{6}$$

It is not entirely clear how to reinterpret the "bundle" from (6) in the noncommutative setting. However, as a minimum, we should have a projection (conditional expectation) from the algebra of "functions on the total space" $C\big(SU_q(3)/\mathbb{T}^2\big)$ onto the algebra of "functions on the base space" $C\big(\mathbb{C}P^2_q\big)$. So we begin by constructing such a conditional expectation.

The algebra $C\big(\mathbb{C}P^2_q\big)$ is a C^*-subalgebra of $C\big(SU_q(3)/\mathbb{T}^2\big)$ in a natural way as follows (cf. [13]). The C^*-subalgebra of $C\big(SU_q(3)\big)$ generated by the first column matrix elements of $\mathbf{u}$, i.e. u_{11}, u_{21} and $u_{\ 1}$, may be identified with the C^*-algebra $C\big(S^5_q\big)$ of continuous functions on the quantum 5-sphere. This C^*-subalgebra is invariant under the gauge action μ of $\mathbb{T}^2$ on $C\big(SU_q(3)\big)$. When restricted to $C\big(S^5_q\big)$, μ reduces to the generator-rescaling circle action $u_{j1} \mapsto z u_{j1}$, $z \quad \mathbb{T}$, whose fixed point algebra is $C\big(\mathbb{C}P^2_q\big)$ (cf. [7,13]). Thus, in the setting of the present article, we have

$$C\big(\mathbb{C}P^2_q\big) = C\big(SU_q(3)/\mathbb{T}^2\big) \cap C^*(u_{11}, u_{21}, u_{\ 1}).$$

In order to construct the desired conditional expectation

$$E : C\big(SU_q(3)/\mathbb{T}^2\big) \to C\big(\mathbb{C}P^2_q\big),$$

we will use integration over a quantum subgroup of $SU_q(3)$ isomorphic to the quantum unitary group $U_q(2)$. Indeed, recall from [10] or [6] that $U_q(2)$ is a compact matrix quantum group with the C^*-algebra of continuous functions $C\big(U_q(2)\big)$ generated densely by three elements u, α, γ, organised into a fundamental unitary matrix

$$\mathbf{v} = \begin{pmatrix} u & 0 & 0 \\ 0 & \alpha & -q\gamma^* u^* \\ 0 & \gamma & \alpha^* u^* \end{pmatrix}.$$

The generator u is central, while $\alpha\gamma = q\gamma\alpha,\ \gamma\gamma^* = \gamma^*\gamma$.

The unitarity of $\mathbf{v}$ implies that u is unitary, while α and γ satisfy the remaining $SU_q(2)$ (cf. [14]) q-commutation rules

$$\alpha\gamma^* = q\gamma^*\alpha, \qquad \alpha^*\alpha + \gamma\gamma^* = 1, \qquad \alpha\alpha^* + q^2\gamma\gamma^* = 1.$$

As shown in [4], the $*$-homomorphism

$$\pi : \mathcal{O}\big(SU_q(3)\big) \longrightarrow \mathcal{O}\big(U_q(2)\big), \qquad \mathbf{u} \mapsto \mathbf{v},$$

is an epimorphism of Hopf algebras, and thus we obtain a right coaction

$$\varrho_{SU_q(\ )} : C\big(SU_q(3)\big) \longrightarrow C\big(SU_q(3)\big) \otimes C\big(U_q(2)\big),\ \ \varrho_{SU_q(\ )} = (\mathrm{id} \otimes \pi) \circ \Delta_{SU_q(\ )}. \quad (7)$$

One immediately checks that

$$\varrho_{SU_q(\ )} \circ \mu_z = (\mu_z \otimes \mathrm{id}) \circ \varrho_{SU_q(\ )},$$

for all $z \quad \mathbb{T}^2$, and this implies that the restriction of $\varrho_{SU_q(\ )}$ to $C\big(SU_q(3)/\mathbb{T}^2\big)$ yields the coaction

$$\varrho_{SU_q(\ )/\mathbb{T}^2} : C\big(SU_q(3)/\mathbb{T}^2\big) \longrightarrow C\big(SU_q(3)/\mathbb{T}^2\big) \otimes C\big(U_q(2)\big).$$

Consequently,

$$(\mathrm{id} \otimes \mathfrak{h}) \circ \varrho_{SU_q(\ )/\mathbb{T}^2} : C\big(SU_q(3)/\mathbb{T}^2\big) \to C\big(SU_q(3)/\mathbb{T}^2\big)^{\mathrm{co}\, U_q(2)}$$

is a faithful conditional expectation. Here $\mathfrak{h}$ denotes the Haar state on $C\big(U_q(2)\big)$ and

$$C(SU_q(3)/\mathbb{T}^2)^{\mathrm{co}\,U_q(2)} = \big\{a \quad C\big(SU_q(3)/\mathbb{T}^2\big) \;:\; \varrho_{SU_q(\ )/\mathbb{T}^2}(a) = a \otimes 1\big\}$$

is the C^*-subalgebra of coinvariants. It is shown in [4] that

$$C\big(SU_q(3)/\mathbb{T}^2\big)^{\mathrm{co}\,U_q(2)} = C\big(SU_q(3)\big) \cap C^*(u_{11}, u_{21}, u_{\ 1}),$$

and thus

$$E = (\mathrm{id} \otimes \mathfrak{h}) \circ \varrho_{SU_q(\ )/\mathbb{T}^2} \tag{8}$$

is the desired faithful conditional expectation from the algebra $C\big(SU_q(3)/\mathbb{T}^2\big)$ onto $C\big(\mathbb{C}P_q^2\big)$.

In order to compute the conditional expectation value (8) it is useful or, indeed necessary, to have an explicit description of the Haar state on $C\big(U_q(2)\big)$. In fact, it is sufficient to have such a description on the dense subalgebra $\mathcal{O}\big(U_q(2)\big)$ of $C\big(U_q(2)\big)$. One way of obtaining the Haar measure is first to realise that $\mathcal{O}\big(U_q(2)\big)$ is a right $\mathcal{O}\big(SU_q(2)\big)$-comodule algebra (i.e. the quantum group $SU_q(2)$ acts on $U_q(2)$) with fixed points equal to $\mathcal{O}\big(U(1)\big)$ and then to compose the Haar integrals on $\mathcal{O}\big(SU_q(2)\big)$ and $\mathcal{O}\big(U(1)\big)$ (both well-known, the first one described by Woronowicz in [15]).

The coaction $\varrho_{U_q(2)}$ of $\mathcal{O}(SU_q(2))$ on $\mathcal{O}(U_q(2))$ is induced from the Hopf-algebra projection

$$\mathcal{O}\big(U_q(2)\big) \xrightarrow{\ p\ }\!\!\!\!\rightarrow \mathcal{O}\big(SU_q(2)\big)\,, \quad \begin{pmatrix} u & 0 & 0 \\ 0 & \alpha & -q\gamma^* u^* \\ 0 & \gamma & \alpha^* u^* \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & -q\gamma^* \\ 0 & \gamma & \alpha^* \end{pmatrix},$$

by

$$\varrho_{U_q(2)} = (\mathrm{id} \otimes p) \circ \Delta_{U_q(2)} : \mathcal{O}\big(U_q(2)\big) \longrightarrow \mathcal{O}\big(U_q(2)\big) \otimes \mathcal{O}\big(SU_q(2)\big). \tag{9}$$

As $*$-algebra the coinvariants of this coaction are generated by the unitary u, and hence are isomorphic to $\mathcal{O}\big(U(1)\big)$. The Haar functional $\mathfrak{h}$ on $\mathcal{O}\big(U_q(2)\big)$ (and, consequently, on $C\big(U_q(2)\big)$) is the composite

$$\mathfrak{h} :\ \mathcal{O}\big(U_q(2)\big) \quad^{\varrho_{U_q(2)}} \quad \mathcal{O}\big(U_q(2)\big) \otimes \mathcal{O}\big(SU_q(2)\big) \quad^{\mathrm{id}\otimes \quad SU_q(2)} \quad \mathcal{O}\big(U(1)\big) \quad^{U(1)} \quad \mathbb{C}.$$

Here $\mathfrak{h}_{SU_q(2)}$ is the Haar measure on the quantum group $SU_q(2)$ given on the standard basis of $\mathcal{O}\big(U_q(2)\big)$ as

$$\mathfrak{h}_{SU_q(2)}\left(\alpha^k \gamma^m \gamma^{*n}\right) = \delta_{k0}\, \delta_{mn}\, \frac{q^2 - 1}{q^{2n+2} - 1}, \quad \text{for all } k \quad \mathbb{Z},\ m, n \quad \mathbb{N}, \tag{10}$$

where we use the convention that, for $k < 0$, $x^k = (x^*)^{-k}$; see [15, Appendix A1]. The Haar functional $\mathfrak{h}_{U(1)}$ on the standard basis of $\mathcal{O}(U(1))$ is given by

$$\mathfrak{h}_{U(1)}\left(u^k\right) = \delta_{k0}, \quad \text{for all } k \quad \mathbb{Z}. \tag{11}$$

Combining formulae (9)–(11) we thus obtain an explicit expression for the Haar functional on $\mathcal{O}(U_q(2))$,

$$\mathfrak{h}\left(\alpha^k u^l \gamma^m \gamma^{*n}\right) = \delta_{k0}\,\delta_{l0}\,\delta_{mn}\,\frac{q^2-1}{q^{2n+2}-1}, \quad \text{for all } k,l \quad \mathbb{Z},\ m,n \quad \mathbb{N}. \tag{12}$$

With the explicit formula (12) at hand we can now compute the value of the conditional expectation (8) on the elements (3) of the quantum flag variety algebra $\mathcal{O}\big(SU_q(3)/\mathbb{T}^2\big)$ densely included in the C^*-algebra of continuous functions $C\big(SU_q(3)/\mathbb{T}^2\big)$. In view of the fact that the coaction $\varrho_{SU_q(\ )/\mathbb{T}^2}$ is the restriction of the map (7) one easily finds that

$$\begin{aligned}
\varrho_{SU_q(\ )}(w_{ijk}) &= w_{ijk}\otimes 1 + u_{i1}u_{j2}u_{k2}\otimes uv_{22}v_2 \\
&\qquad + u_{i1}\left(u_{j\ } u_{k2} + q u_{j2} u_{k\ }\right)\otimes uv_{\ 2}v_{2\ } + u_{i1}u_{j\ }u_{k\ }\otimes uv_{\ 2}v \\
&= w_{ijk}\otimes 1 - q u_{i1}u_{j2}u_{k2}\otimes \alpha\gamma^* \\
&\qquad - q u_{i1}\left(u_{j\ } u_{k2} + q u_{j2} u_{k\ }\right)\otimes \gamma\gamma^* + u_{i1}u_{j\ }u_{k\ }\otimes \gamma\alpha^*.
\end{aligned}$$

Now, the application of $\mathrm{id}\otimes\mathfrak{h}$ together with the commutation rules (1) yield

$$E(w_{ijk}) = \frac{w_{ijk} - w_{ikj}}{1+q^2}.$$

4. Conclusions

In this short note we have studied representations and the structure of the algebra of continuous functions on the quantum flag manifold $SU_q(3)/\mathbb{T}^2$ obtained as the fixed points of the gauge action of the classical two-torus on the quantum $SU(3)$-group. We have also indicated that the quantum flag manifold $SU_q(3)/\mathbb{T}^2$ can be interpreted as the total space of a quantum sphere bundle over the quantum projective space $\mathbb{C}P^2_q$, and we have presented an explicit formula for a faithful conditional expectation from $C\big(SU_q(3)/\mathbb{T}^2\big)$ onto $C\big(\mathbb{C}P^2_q\big)$. The detailed analysis of this bundle is presented in [4].

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A Hopf algebra without a modular pair in involution

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Abstract. The aim of this short note is to communicate an example of a finite-dimensional Hopf algebra that does not admit a modular pair in involution in the sense of Connes and Moscovici.

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1. Introduction

The concept of a modular pair in involution has been introduced by Connes and Moscovici [2] in order to define the Hopf-cyclic cohomology of a Hopf algebra H over a field k. In the following we freely use standard notation from Hopf algebra theory e.g. as in [5, 6]. In particular, H° is the Hopf dual of H and $\beta^{-1} = \beta \circ S$ is the convolution inverse of a group-like $\beta \quad H^\circ$ (i.e. a character $\beta \colon H \to k$).

Definition. Let H be a Hopf algebra. A pair (l, β) of group-like elements $l \quad H, \beta \quad H^\circ$ is a *modular pair in involution* if $\beta(l) = 1$ and

$$S^2(h) = \beta(h_{(1)}) l h_{(2)} l^{-1} \beta^{-1}(h_{(\ )}) \tag{1}$$

holds for all $h \quad H$.

Hajac et al. extended this notion to that of stable anti Yetter–Drinfel'd modules over Hopf algebras [3]. It is also related to earlier work by Kauffman and Radford [4] who classified the ribbon elements in Drinfel'd doubles of finite-dimensional Hopf algebras. Among their results they showed that if $\dim_k H$ is odd and S^2 has

We thank P.M. Hajac for pointing us to the question answered here.

odd order, then there is always a pair (l, β) implementing S^2 as in (1). The question arises whether there are also always pairs (l, β) that additionally satisfy the stability condition $\beta(l) = 1$. The aim of the present note is to point out that this is not the case in general:

Theorem. *Let p be a prime number, $s \quad \mathbb{Z}_p \setminus \{0\}$, $q \quad k$ be a primitive pth root of unity, and H be the Hopf algebra with generators g, x, y and defining algebra and coalgebra relations*

$$gx = qxg, \quad gy = q^{-s}yg, \quad g^p = 1, \quad x^p = y^p = 0, \quad xy = q^{-s}yx,$$
$$\Delta(g) = g \otimes g, \quad \Delta(x) = 1 \otimes x + x \otimes g, \quad \Delta(y) = 1 \otimes y + y \otimes g^s.$$

Its antipode is determined by

$$S(g) = g^{-1}, \qquad S(x) = -xg^{-1}, \qquad S(y) = -yg^{-s},$$

and H has a modular pair in involution if and only if $s \quad \{1, p-1\}$.

The Hopf algebra H appears naturally in several contexts. In particular, it is referred to as the book Hopf algebra in [1].

2. Proof

It is immediately verified that the group-likes in H are the elements of the form $l = g^i$ for some i. Furthermore, a character $\beta\colon H \to k$ has to vanish on x, y and is determined by its value $\beta(g)$ which can be any pth root of unity in k (including 1, in which case $\beta = \varepsilon$ is the counit of H). It follows that

$$T\colon H \to H, \qquad h \mapsto \beta(h_{(1)}) l h_{(2)} l^{-1} \beta^{-1}(h_{(\ )})$$

is the Hopf algebra automorphism of H determined by

$$T(g) = g, \qquad T(x) = q^i \beta(g)^{-1} x, \qquad T(y) = q^{-is} \beta(g)^{-s} y.$$

Comparing this with

$$S^2(g) = g, \qquad S^2(x) = gxg^{-1} = qx, \qquad S^2(y) = g^s y g^{-s} = q^{-s^2} y$$

shows that $S^2 = T$ if and only if

$$\beta(g) = q^{i-1}, \qquad \beta(g)^s = q^{(s-i)s}.$$

Assuming $\beta(g) = q^{i-1}$, we obtain

$$\beta(l) = \beta(g)^i = q^{i(i-1)}$$

and the condition $\beta(g)^s = q^{(s-i)s}$ reduces to $q^{(1-2i+s)s} = 1$. Thus $S^2 = T$ holds if and only if we have

$$(1 - 2i + s)s = 0 \quad \mathbb{Z}_p.$$

As $s \neq 0$, this is equivalent to

$$s = 2i - 1 \quad \mathbb{Z}_p.$$

In total we see that (l, β) is a modular pair in involution if and only if

$$s = 2i - 1, \quad i(i-1) = 0 \quad \mathbb{Z}_p.$$

For $i = 0$ this means $s = -1 = p - 1$ and for $i = 1$ it means $s = 1$ in $\mathbb{Z}_p$. The claim follows. □

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Part IV

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Hopf–Rinow theorem in Grassmann manifolds of C^*-algebras

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Abstract. We survey several results on the problem of finding a geodesic of minimal length joining two given endpoints in Grassmann manifolds of C^*-algebras.

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Keywords. Grassmann manifold, minimal geodesic, C^*-algebra, compact operator, restricted Grassmannian.

1. Introduction

In the context of finite-dimensional Riemannian manifolds, the Hopf–Rinow theorem is the main result establishing sufficient conditions to find a geodesic of minimal length joining two given points in the manifold. It is stated as follows:

Theorem (Hopf–Rinow). *Let X be a finite-dimensional connected Riemannian manifold, and denote by d the associated geodesic distance. If (X, d) is a complete metric space, then there exists a geodesic of minimal length joining any two points in X.*

We refer to Lang's book [18] for a proof, as well other equivalent conditions to geodesic completeness. It turns out that the Hopf–Rinow theorem carries over to finite-dimensional Finsler manifolds [7], where the metric is given by a norm on each tangent space, which does not necessarily come from an inner product. Further, it was also generalized to locally compact metric spaces by S. Cohn-Vossen [10, 15]. At this point, let us mention that local compactness is the crucial hypothesis in all of these versions. This is explicitly assumed in metric spaces, while in the other cases is deduced from the fact that the manifolds considered are finite dimensional.

In this work we are interested in infinite dimensional manifolds, where as one might expect by the lack of local compactness, the situation completely changes. Even for Hilbert–Riemannian manifolds, that is, manifolds modeled on a Hilbert space, Hopf–Rinow theorem is false. More precisely, there are examples of complete Hilbert–Riemannian manifolds such that there exist two points that cannot be joined by a minimal geodesic [15, 20], or even worse, cannot be joined by a geodesic [6]. A local result still holds true in this setting: any two points inside a normal neighborhood can be joined by a minimal geodesic.

The aim of this work is to present the main results on the minimality of geodesics in Grassmann manifolds of C^*-algebras. The Grassmann manifold of a unital C^*-algebra $\mathcal{A}$ is defined as the set of all orthogonal projections in $\mathcal{A}$. Thus, there is a natural norm at hand, given by the spectral norm of bounded operators, to put on tangent spaces. An obvious fact, but worth to remark, is that this defines a non-Riemannian metric. It is not difficult to show that geodesic distance associated to the metric gives a complete metric space. The question is then to know if the Hopf–Rinow theorem holds in this setting. Since our Grassmann manifolds are homogeneous spaces, the results exhibited here can be seen as examples of a more general topic, the metric geometry of infinite dimensional homogeneous spaces arising in operator theory (see [13, 14, 19]).

2. The Grassmann manifold of a C^*-algebra

Throughout, $\mathcal{A}$ denotes a unital C^*-algebra. We shall think that $\mathcal{A}$ is contained in the algebra of bounded operators of some Hilbert space; thus the norm $\|\cdot\|$ of the algebra $\mathcal{A}$ is the spectral norm of operators. This section is dedicated to explaining general geometric features of Grassmann manifolds of C^*-algebras.

Definition 1. The *Grassmann manifold of* $\mathcal{A}$ is given by

$$Gr(\mathcal{A}) = \{\, P \in \mathcal{A} \,:\, P = P^* = P^2 \,\}.$$

Remark 2. It'll be important to remember that $\|P - Q\| \leq 1$, when $P, Q \in Gr(\mathcal{A})$.

The unitary group $\mathcal{U}_{\mathcal{A}}$ of $\mathcal{A}$ acts on $Gr(\mathcal{A})$ as follows: $U \cdot P = UPU^*$, $U \in \mathcal{U}_{\mathcal{A}}$, $P \in Gr(\mathcal{A})$. This action has the following properties:

(i) It is locally transitive: if $\|P - Q\| < 1$, then there is a unitary $U \in \mathcal{U}_{\mathcal{A}}$ such that $UPU^* = Q$.

(ii) The isotropy group at $P \in Gr(\mathcal{A})$, i.e. $\mathcal{I}_P = \{\, U \in \mathcal{U}_{\mathcal{A}} \,:\, UP = PU \,\}$ is a Banach–Lie subgroup of $\mathcal{U}_{\mathcal{A}}$.

(iii) The orbit of $P \in Gr(\mathcal{A})$, namely $\mathcal{O}_P = \{\, UPU^* \,:\, U \in \mathcal{U}_{\mathcal{A}} \,\}$ contains the connected components of $Gr(\mathcal{A})$.

In fact, the third item can be derived from the first. For we shall use that the orbits have manifold structure endowed with the ambient topology (this is independently discussed below), and that path components and connected components coincide for manifolds. We take a continuous curve $\gamma : [0, 1] \to Gr(\mathcal{A})$ such that

$\gamma(0) = P$ and $\gamma(1) = Q$. Then there exist points $0 = t_0 < t_1 < \cdots < t_n = 1$ satisfying $\|\gamma(t_i) - \gamma(t_{i-1})\| < 1$, so that $\gamma(t_i) = U_i\gamma(t_{i-1})U_i^*$ for some $U_i \in \mathcal{U}_{\mathcal{A}}$. Then $Q = U_n \cdots U_1 P(U_n \cdots U_1)^*$. In the case where $\mathcal{A} = \mathcal{B}(\mathcal{H})$, i.e. the bounded operators on a Hilbert space $\mathcal{H}$, it holds that connected components coincide with orbits. Moreover, they are parameterized by the rank and corank of the projections.

On the other hand, the second item above is the sufficient condition to guarantee the following result (see e.g. [8, Thm. 4.9]).

Proposition 3. *The orbit $\mathcal{O}_P \simeq \mathcal{U}_A/\mathcal{I}_P$ is a real analytic manifold endowed with the quotient topology, and the map $\pi_P : \mathcal{U}_{\mathcal{A}} \to \mathcal{O}_P$, $\pi_P(U) = UPU^*$, is a real analytic submersion.*

Noting that $Gr(\mathcal{A})$ can be expressed as a disjoint union of orbits, we then have that $Gr(\mathcal{A})$ endowed with the quotient topology has manifold structure. Let $\mathcal{A}_{sa}$ be the self-adjoint operators in $\mathcal{A}$. H. Porta and L. Recht proved:

Proposition 4 ([21]). *$Gr(\mathcal{A})$ is a real analytic submanifold of $\mathcal{A}_{sa}$.*

This means that there is an adapted coordinate chart around each projection in $Gr(\mathcal{A})$, or equivalently by a criterion in [9], that the quotient topology on $Gr(\mathcal{A})$ coincides with topology inherited from $\mathcal{A}_{sa}$, and tangent spaces are closed and complemented in $\mathcal{A}_{sa}$. It is worth pointing out that the differential structures given in Proposition 3 and 4 coincide in $Gr(\mathcal{A})$.

Remark 5. The tangent space at $P \in Gr(\mathcal{A})$ can be computed as

$$\big(TGr(\mathcal{A})\big)_P = \{\, XP - PX \,:\, X^* = -X \in \mathcal{A}\,\}.$$

To see this, just take the derivative of the curve $\gamma(t) = e^{tX}Pe^{-tX}$ at $t = 0$ to prove one inclusion. For the reversed inclusion, let $\gamma : (-\epsilon, \epsilon) \to Gr(\mathcal{A})$ be a smooth curve satisfying $\gamma(0) = P$, and set $Y = \dot\gamma(0) \in \mathcal{A}_{sa}$. Using that $\gamma(t)^2 = \gamma(t)$, and taking the derivative at $t = 0$, we find that $YP + PY = Y$. This implies that Y is P-codiagonal, i.e. $PYP = (I - P)Y(I - P) = 0$. Now set $X = YP - PY$, which satisfies $Y = XP - PX$.

Remark 6. From the previous remark, it is straightforward to give a (continuous) projection onto each tangent space. Using a block decomposition in terms of the projection P, we can rewrite the tangent space as

$$\big(TGr(\mathcal{A})\big)_P = \left\{ \begin{pmatrix} 0 & x_{12} \\ x_{12}^* & 0 \end{pmatrix} \,:\, x_{12} \in P\mathcal{A}(I - P) \right\}.$$

Then we can define the projection

$$\mathcal{E}_P : \mathcal{A}_{sa} \to \big(TGr(\mathcal{A})\big)_P\,, \quad \mathcal{E}_P\left(\begin{pmatrix} x_{11} & x_{12} \\ x_{12}^* & x_{22} \end{pmatrix}\right) = \begin{pmatrix} 0 & x_{12} \\ x_{12}^* & 0 \end{pmatrix}.$$

3. Geodesics of minimal length

Following [21], we now define a linear connection using the projection onto tangent spaces introduced in the previous section. Let X be a smooth tangent field along a curve γ in $Gr(\mathcal{A})$, then the linear connection is

$$\frac{DX}{Dt} = \mathcal{E}_{\gamma(t)}(\dot{\gamma}(t)).$$

Remark 7. The unique geodesic δ satisfying the initial conditions $\gamma(0) = P$, $\dot{\gamma}(0) = \begin{pmatrix} 0 & x_{12} \\ x_{12}^* & 0 \end{pmatrix}$, can be explicitly computed:

$$\delta(t) = e^{t\begin{pmatrix} 0 & -x_{12} \\ x_{12}^* & 0 \end{pmatrix}} P e^{-t\begin{pmatrix} 0 & -x_{12} \\ x_{12}^* & 0 \end{pmatrix}}.$$

Now given a curve smooth $\gamma : [0,1] \to Gr(\mathcal{A})$, we define its length by

$$L(\gamma) = \int_0^1 \|\dot{\gamma}(t)\| \, dt.$$

Note that this norm is not smooth or strictly convex. Thus, the tools of calculus of variations cannot be used to find the extremals of the length functional. The geodesic distance between points P, Q in the same connected component is defined by $d(P,Q) = \inf\{\, L(\gamma) \;:\; \gamma$ piecewise smooth curve joining P and $Q\,\}$. A curve $\gamma \subseteq Gr(\mathcal{A})$ joining P and Q has minimal length if $L(\gamma) = d(P,Q)$. Geodesic completeness is not difficult to prove.

Lemma 8. *$(Gr(\mathcal{A}), d)$ is a complete metric space.*

Proof. Pick a sequence $(P_n) \subseteq Gr(\mathcal{A})$ such that $d(P_n, P_m) \to 0$. Using that straight lines are shortest paths in any vector space, we have the estimate $\|P_n - P_m\| \leq d(P_n, P_m)$. Then there is a projection $P_0 \quad Gr(\mathcal{A})$ satisfying $\|P_n - P_0\| \to 0$. Recalling that the action is locally transitive, we can write $P_n = U_n P_0 U_n^*$ for some unitaries U_n and $n \geq 1$ large enough. By Proposition 3, the map π_{P_0} is a submersion, thus in particular has continuous local cross sections. Since the quotient topology and the ambient topology coincide, we obtain that $\|U_n - I\| \to 0$. Next put $U_n = e^{X_n}$ for some $X_n \quad \mathcal{A}$, $X_n = -X_n^*$. Therefore $d(P_n, P_0) \leq L(\gamma_n) = \|X_n P_0 - P_0 X_n\| \to 0$, where $\gamma_n(t) = e^{tX} P_0 e^{-tX_n}$. □

The rest of this section is devoted to analyze the validity of the Hopf–Rinow theorem, or equivalently, to find geodesics of minimal length. We first state the following local result.

Theorem 9 ([21]). *Let $P, Q \quad Gr(\mathcal{A})$ such that $\|P - Q\| < 1$. Then there exists a unique geodesic of minimal length joining P and Q.*

Let us mention that the proof relies on a beautiful geometric idea: compare the length of curves in the Grassmann manifold with curves in the sphere of a Hilbert space. Now we outline some ideas in the proof. The estimate $\|P - Q\| < 1$ is used to find a P-codiagonal operator such that $X \quad \mathcal{A}$, $X = -X^*$, $\|X\| < \pi/2$

and $e^X P e^{-X} = Q$. Moreover, the operator X depends analytically on P. It is found by passing from the projections P, Q to the symmetries $\epsilon_P = 2P - I$, $\epsilon_Q = 2Q - I$, and using the property that the exponential map

$$\exp : \{ X \quad \mathcal{A} : X = -X^*, \|X\| < \pi \} \to \{ U \quad \mathcal{U}_{\mathcal{A}} : \|U - I\| < 2 \}$$

is an (analytic) diffeomorphism. An application of the GNS representation yields a length-reducing map $F : Gr(\mathcal{A}) \to \mathcal{S}_{\mathcal{H}}$, where $\mathcal{S}_{\mathcal{H}}$ is the sphere of a Hilbert space $\mathcal{H}$. This map also satisfies the condition that the geodesic $\delta(t) = e^{tX} P e^{-tX}$ is mapped isometrically onto a geodesic of $\mathcal{S}_{\mathcal{H}}$. Now take γ a curve in $Gr(\mathcal{A})$ joining P and Q, we have the following estimates:

$$L(\delta) = L(F\delta) \leq L(F\gamma) \leq L(\gamma),$$

where in the second inequality we have used the fact that geodesics in the sphere are minimizing up to π.

In the case in which $\mathcal{A} = \mathcal{B}(\mathcal{H})$, we simple write $Gr(\mathcal{H}) = Gr(\mathcal{B}(\mathcal{H}))$. For this case, E. Andruchow proved the following equivalent conditions.

Theorem 10 ([1]). *Given $P, Q \quad Gr(\mathcal{H})$, the following are equivalent:*

(i) *There is a minimal geodesic joining P and Q.*
(ii) *There is a geodesic joining P and Q.*
(iii) $\dim(\operatorname{ran}(P) \cap \ker(Q)) = \dim(\ker(P) \cap \operatorname{ran}(Q))$.

Moreover, there is unique minimal geodesic if and only if the above dimensions are zero.

The proof depends on ideas of P. Halmos [17], J. Dixmier [12] and C. Davis [11]. They independently proposed that to understand the geometry of two subspaces $\mathcal{S} = \operatorname{ran}(P)$, $\mathcal{T} = \operatorname{ran}(Q)$ ($\mathcal{S}^\perp = \ker(P)$, $\mathcal{T}^\perp = \ker(Q)$), one needs to decompose the Hilbert space as

$$\mathcal{H} = (\mathcal{S} \cap \mathcal{T}) \oplus (\mathcal{S}^\perp \cap \mathcal{T}^\perp) \oplus (\mathcal{S}^\perp \cap \mathcal{T}) \oplus (\mathcal{S} \cap \mathcal{T}^\perp) \oplus \mathcal{H}_0 \,,$$

where $\mathcal{H}_0$ is defined as the orthogonal complement to the first four summands, and it is known as the generic part of the two subspaces. For instance, the key step to prove the implication (iii) $\Longrightarrow$ (i) is to find a P-codiagonal operator X such that $X = -X^*$, $\|X\| \leq \pi/2$ and $e^X P e^{-X} = Q$. Once this is obtained, the proof follows the same argument as in Theorem 9. To find an operator X satisfying the mentioned conditions, one notes that each of the five subspaces is invariant for both P and Q. It is then possible to use a special representation of the projections on $\mathcal{H}_0$ given by Halmos to find X on $\mathcal{H}_0$; on the first and second summands the problem is trivial since P and Q act as the identity and zero, respectively; and the existence of X restricted to third and fourth summands is deduced from the hypothesis on the dimensions.

Our last results concern the Grassmann manifold of a particular C^*-algebra studied in collaboration with E. Andruchow and M.E. Di Iorio y Lucero [2]. Let $\mathcal{H}$ be a separable Hilbert space such that $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where $\mathcal{H}_+$ and $\mathcal{H}_-$ are both infinite dimensional closed subspaces. Denote by E_+ (resp. E_-) the orthogonal projection onto $\mathcal{H}_+$ (resp. $\mathcal{H}_-$). Let $\mathcal{K}(\mathcal{H})$ be the algebra of compact operators on $\mathcal{H}$. Next we consider the C^*-algebra of operators with compact commutator, i.e.

$$\mathcal{B}_{cc} = \{ X \quad \mathcal{B}(\mathcal{H}) : XE_+ - E_+X \quad \mathcal{K}(\mathcal{H}) \}.$$

Writing an operator

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$$

as a block operator in terms of the projections E_+ and E_-, shows that $X \quad \mathcal{B}_{cc}$ if and only if x_{12}, x_{21} are compact. We denote by $\mathcal{U}_{cc}$ the unitary group of $\mathcal{A}_{cc}$. For a unitary

$$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \quad \mathcal{U}_{cc}\,,$$

u_{11}, u_{22} are Fredholm operators, their indices are related by $\operatorname{ind}(u_{11}) = -\operatorname{ind}(u_{22})$, and they determine the connected components of $\mathcal{U}_{cc}$. The elements of the corresponding Grassmann manifold $Gr(\mathcal{B}_{cc})$ will be called *essentially commuting projections* (with respect to the decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$).

We denote by $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H}) := \mathcal{C}(\mathcal{H})$ the projection onto the Calkin algebra, and set $e_\pm = \pi(E_\pm)$. Then there are exactly nine classes of essentially commuting projections, which can be defined by means of the matrix representation in terms of e_+ and e_- of the images of the projections under π. On the one hand, the *discrete classes* $\mathbb{D}_i$, $i = 1, \dots,$, are given by

$$0,\ 1,\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

On the other hand, the *essential classes* $\mathbb{E}_i$, $i = 1, \dots, 5$, are

$$\begin{pmatrix} p_+ & 0 \\ 0 & 0 \end{pmatrix},\ \begin{pmatrix} p_+ & 0 \\ 0 & 1 \end{pmatrix},\ \begin{pmatrix} 0 & 0 \\ 0 & p_- \end{pmatrix},\ \begin{pmatrix} 1 & 0 \\ 0 & p_- \end{pmatrix},\ \begin{pmatrix} p_+ & 0 \\ 0 & p_- \end{pmatrix},$$

where p_+, p_- are proper projections in $\mathcal{C}(\mathcal{H})$.

A projection P belongs to $\mathbb{D}_1$ if and only if $\pi(P) = 0$. This means that P is compact, so it must have finite rank. Similarly, projections in $\mathbb{D}_2$ have finite corank. The connected components of the class $\mathbb{D}_1$ (resp. $\mathbb{D}_2$) are parameterized by the rank (resp. corank). For the other two discrete classes, we need to recall the following:

Definition 11. An orthogonal projection P belongs to the *restricted Grassmannian* $Gr_{\ es}(\mathcal{H}_+)$ if

(i) $E_+P|_{\operatorname{ran}(P)} : \operatorname{ran}(P) \to \mathcal{H}_+$ is Fredholm,
(ii) $E_-P|_{\operatorname{ran}(P)} : \operatorname{ran}(P) \to \mathcal{H}_-$ is compact.

This Grassmann manifold was first studied due to its connections with KdV equations and loop groups [22, 23]. It turns out that class $\mathbb{D}$ coincides with $Gr_{\ es}(\mathcal{H}_+)$, while class $\mathbb{D}_4$ is given by $Gr_{\ es}(\mathcal{H}_-)$. It is a well-known fact that the index of the operator $E_+P|_{\operatorname{ran}(P)}$ gives the connected component where P lies.

Theorem 12 ([2]). *Let P, Q two projections in the same connected component of a discrete class $\mathbb{D}_i$, $i = 1, \ldots,$. Then there is a geodesic of minimal length joining P and Q. Furthermore, it is unique if and only if* $\ker(P) \cap \operatorname{ran}(Q) = \operatorname{ran}(P) \cap \ker(Q) = \{0\}$.

Remark 13. The proof idea for classes $\mathbb{D}$ and $\mathbb{D}_4$ follows the method of Theorem 10, and also the main minimality result in [4], where the restricted Grassmannian associated to the Hilbert–Schmidt operators was studied instead of the compact operators. It is surprising that, using a completely different technique, the minimality of geodesics also holds for any other symmetrically-normed ideal [5]. For the ideal of compact operators, it gives our result for the classes $\mathbb{D}$ and $\mathbb{D}_4$.

In contrast, essential classes behave completely different.

Theorem 14 ([2]). *The following assertions hold:*

(i) *The sets $\mathbb{E}_i$, $i = 1, \ldots, 5$, are connected, and the action of $(\mathcal{U}_{cc})_0$, the connected component of the identity, is transitive on each $\mathbb{E}_i$.*

(ii) *The Hopf–Rinow theorem is not valid on each $\mathbb{E}_i$, $i = 1, \ldots, 5$.*

It is not difficult to construct an example for (ii). We give it for class $\mathbb{E}_1$; similar examples can be given for the other classes. Take the following projections in $\mathbb{E}_1$:

$$P = \begin{pmatrix} p_+ & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} q_+ & 0 \\ 0 & 0 \end{pmatrix},$$

such that $\dim(\operatorname{ran}(q_+) \cap \ker(p_+)) \neq \dim(\operatorname{ran}(p_+) \cap \ker(q_+))$. By Theorem 10, we find that there is no minimal geodesic in $Gr(\mathcal{H})$. Using the particular form of the geodesics, this clearly implies that there is no minimal geodesic in $Gr(\mathcal{B}_{cc})$.

Remark 15. Concrete examples of essentially commuting projections onto shift-invariant subspaces are given in [3].

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Short geodesics for Ad invariant metrics in locally exponential Lie groups

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Abstract. We study the geodesic structure of Lie groups K that admit bi-invariant metrics: the main results concern the fact that one-parameter groups are short paths for those metrics, as in the Riemannian case.

Mathematics Subject Classification (2000). Primary 22E65, 58B20; Secondary 53C22, 47B10, 58D05.

Keywords. Lie group, Banach space, Finsler metric, bi-invariant metric, geodesic, one-parameter group, isometry group, unitary group.

1. Introduction

The purpose of this paper is to present a systematic framework to the theory of intrinsic distances derived from tangent metrics, in the setting of Lie groups with bi-invariant metrics. The emphasis is on infinite-dimensional Lie groups, modeled by locally convex topological vector spaces, and metrics defined as continuous tangent Finsler norms (all the metrics considered in this paper are non-negative).

There are technical and fundamental differences with the classical theory of metrics on Lie groups, coming from two directions: the first one is going from finite-dimensional manifolds to infinite-dimensional manifolds, and the second one is going from Riemannian (or classical Finsler metrics, which are smooth and have an auxiliary Riemannian metric given in terms of the Hessian of the metric) to only continuous tangent metrics. However, it is more often than not that both difficulties arise simultaneously, since infinite-dimensional vector spaces come with intrinsic and interesting continuous norms or semi-norms.

When the metric is not Riemannian, and there is no connection or distribution: which is the right notion of geodesic, in the absence of Euler's equations? We have taken the approach of the metric geometry (or length spaces, see Burago-Burago and Ivanov [7]): a path in a manifold is a geodesic if it is minimizing, that is, if its length equals the distance among the endpoints. The distance, of course, is defined as the infimum of the lengths of paths joining given endpoints (usually called the *rectifiable distance*).

If a group G admits a bi-invariant Riemannian metric g, it is well-known that geodesics δ of (G, g) are left-translations of one-parameter groups: $\delta(t) = ue^{tv}$ (cf. [10, 2.90]). If the metric is not Riemannian, it is also expected that one-parameter groups will be short paths for the rectifiable distance, and this is the theorem stated in Section 2 of the paper. We also claim that when the norm is strictly convex, those are the unique possible short paths. The main results are then Theorems 13 and 16.

1.1. Lie groups and rectifiable metrics

Let us present in this section some general definitions and considerations that will be used throughout the paper.

Manifolds in this paper will be modeled with charts in a Hausdorff locally convex topological vector space (shortly l.c.s.). The differential of a map $f : M \to N$ among smooth manifolds will be denoted by $f_* : TM \to TN$ and its specialization by $f_{*p} : T_pM \to T_{f(p)}N$, $p \in M$.

In this paper, a *Lie group* G is a manifold such that the operation $(x, y) \mapsto xy^{-1}$ is smooth (at least C^2) as a map $G \times G \to G$. If $g \in G$ and $L_g : h \mapsto gh$ denotes the left multiplication in G, with some abuse of notation we denote

$$gv = L_g v = (L_g)_{*h} v \in T_{gh}G$$

for $h \in G$, $v \in T_hG$. We denote $1 \in G$ the identity of the group and $\mathrm{Lie}\,(G) = T_1G$ its Lie algebra. The Lie bracket in $\mathrm{Lie}\,(G)$ will be denoted by $[\cdot, \cdot]$: it is always a bi-linear, anti-symmetric and continuous map. If $c_g(h) = ghg^{-1}$ is the conjugation automorphism, i.e. $c_g = L_g R_g^{-1}$ for $g \in G$, we follow the standard notation $\mathrm{Ad}_g = (c_g)_{*1}$ with $\mathrm{Ad} : G \to \mathrm{GL}(\mathrm{Lie}\,(G))$ a group homomorphism.

If $\mathrm{Lie}\,(G)$ is not a Banach space then $\mathrm{GL}(\mathrm{Lie}\,(G))$ is not necessarily a Lie group, but it is a subgroup of the space of diffeomorphisms of $\mathrm{Lie}\,(G)$ therefore there is a natural notion of smoothness. We denote

$$\mathrm{ad} = (\mathrm{Ad})_{*1} : \mathrm{Lie}\,(G) \to \mathcal{L}(\mathrm{Lie}\,(G))$$

which is a linear Lie algebra morphism, and in fact $\mathrm{ad}(v)(w) = [v, w]$ for any $v, w \in \mathrm{Lie}\,(G)$ (see Neeb [17, Section II.3]).

1.1.1. Finsler metrics. In this section we define continuous Finsler metrics in Lie groups. For a more radical approach that drops the smoothness assumption on the manifolds, see Andreev [1]. See also Berestovskii [4, 5] for a systematic account of finite-dimensional homogeneous manifolds with Finsler metrics defined by distributions in the fiber bundle.

Definition 1 (Finsler norms and semi-norms). Let E be a l.c.s., $\mu = |\cdot| : E \to \mathbb{R}_{\geq 0}$ a continuous function. Then μ is a *Finsler norm* if it is sub-additive and positively homogeneous: $|v + w| \leq |v| + |w|$ and $|\lambda v| = \lambda |v|$ for $v, w \quad E$ and $\lambda \quad \mathbb{R}_{\geq 0}$, and $|v| = 0$ implies $v = 0$.

If $|tv| = |t|\,|v|$ for all $t \quad \mathbb{R}$, we obtain the standard notion of continuous vector space norm.

A word of caution: as discussed in the introduction, our definition of Finsler metric is far more general than the standard one; we are not assuming smoothness, therefore the usual machinery of Riemann–Finsler geometry [3] is not at hand.

Remark 2 (Left-invariant and bi-invariant metrics). If we fix $|\cdot|$ a Finsler norm in $\mathrm{Lie}\,(G)$ and define $|v|_g = \left|(L_g)_{*1}^{-1} v\right|$ for $v \quad T_gG$, then the group G has a *left-invariant Finsler metric* $|\cdot|_g : T_gG \to \mathbb{R}_{\geq 0}$, because if $g, h \quad G$ then $|hv|_{gh} = \left|(gh)^{-1} hv\right| = \left|g^{-1} v\right| = |v|_g$ for $v \quad T_gG$, and the map $(g, v) \mapsto |v|_g = \left|g^{-1}v\right|$ is continuous as a map from TG to $\mathbb{R}$. Any left-invariant Finsler metric in G can be obtained with this procedure. Note also that $\left|\,\mathrm{Ad}_g\, v\right|_1 = \left|vg^{-1}\right|_{g^{-1}} = |v|_1$ when the metric is also right-invariant. In that case we say that the metric is *bi-invariant.* These are the only metrics we will consider in this paper.

Definition 3 (Rectifiable paths and length). We say that a curve $\alpha : [a, b] \to G$ is *rectifiable* if α is differentiable a.e. in some chart of G and $t \mapsto \left|\dot\alpha(t)\right|_{\alpha(t)}$ is Lebesgue integrable. For piecewise smooth or rectifiable arcs $\alpha : [a, b] \to G$, define the *length* of α as

$$\mathrm{Length}_\mu(\alpha) = \int_a^b \left|\dot\alpha(t)\right|_{\alpha(t)} dt.$$

Definition 4. For $g, h \quad G$, consider the infimum of the lengths of such arcs joining g, h in G,

$$\mathrm{dist}_\mu(g, h) = \inf \left\{ \mathrm{Length}_\mu(\alpha) : \alpha : [0, 1] \to G \text{ rectifiable}, \alpha(0) = g, \alpha(1) = h \right\}.$$

Then $\mathrm{dist}_\mu : G \times G \to \mathbb{R}_{\geq 0}$ is a p.s.d. (pseudo-quasi-distance): it is finite in each arc-wise connected component of G, it obeys the triangle inequality, and it is reversible if and only if $|\cdot|_g$ is homogeneous (if it is a norm) for each $g \quad G$. More details on asymmetric distances can be found in [14] and the references therein.

Remark 5. Whether $\mathrm{dist}_\mu(x, y) = 0$ implies $x = y$ in G is more subtle. There are examples when this fails, see Michor and Mumford [15] for such an example; see also the paper by Clarke [8].

A sufficient condition to obtain the non-degeneracy of dist_μ (for left invariant metrics) is given by asking $|\cdot|$ to induce in $\mathrm{Lie}\,(G)$ its original l.c.s topology. In particular, if G is a finite-dimensional Lie group, dist_μ is non-degenerate for any chosen *norm* in $\mathrm{Lie}\,(G)$.

Definition 6. We will denote with (G, dist_μ) the underlying (pseudo-quasi) metric space. Nevertheless, this distance or quasi-distance induces a topology in G, and we will refer to the topology induced as τ_μ when needed; otherwise the topology of G will always be the manifold topology denoted by τ_G. Clearly, τ_μ will be Hausdorff if and only if dist_μ is non-degenerate. It is apparent that $\mathrm{dist}_\mu : (G, \tau_G) \times (G, \tau_G) \to \mathbb{R}$ is continuous, and τ_μ is finer than τ_G.

Remark 7. If G carries a smooth exponential function, then its derivative can be computed explicitly: for $v, w \quad \mathrm{Lie}\,(G)$,

$$\exp_{*w}(v) = e^w \int_0^1 \mathrm{Ad}_{e^{-sw}}\, v\, ds = e^w \int_0^1 e^{-s\ \mathrm{ad}\, w} v\, ds.$$

2. Main results

In this section we establish the main results of the paper, with some background and context. For proofs, see [13]. These results extend those obtained for groups of compact operators [2] with symmetric norms.

We assume that the Lie group K has a smooth (at least C^2) exponential map. Thus for fixed $v \quad \mathrm{Lie}\,(K)$, $w \quad \mathrm{Lie}\,(K)$, the path $f(s) = e^{s\,\mathrm{ad}\,w} v = \mathrm{Ad}_{e^{sw}}\, v$ is smooth.

The next lemma contains essentially the same information as the Gauss' Lemma of Riemannian geometry: the differential of the exponential map along a geodesic preserves angles with the geodesic speed vector.

Lemma 8 (Gauss' Lemma). *Let $v, w \quad \mathrm{Lie}\,(K)$ and φ be a norming functional for w, $\varphi(w) = |w|$. Then*

1. $\varphi(e^{\lambda\,\mathrm{ad}\,w} v) = \varphi(v)$ *for all* $\lambda \quad \mathbb{R}$,
2. $\varphi(e^{-w} \exp_{*w} v) = \varphi(v)$.

Since they will play a fundamental role in bi-invariant metrics, let's define segments and polygonal paths:

Definition 9. A path δ in K is a *segment* if it is a left translation of a one-parameter group, i.e. $\delta(t) = ue^{tz}$. We say that δ *polygonal path* if it is a continuous concatenation of segments.

This is the main result of this section, that follows from Gauss lemma:

Lemma 10. *Let $w : [a, b] \to \mathrm{Lie}\,(K)$ be a piecewise C^1 path, let $\gamma = e^w$. Then*

$$|w(b)| - |w(a)| \leq \int_a^b \left|\gamma_t^{-1} \dot{\gamma}_t\right| dt = \mathrm{Length}_\mu(\gamma).$$

Thus when w starts at 0 (equivalently, when γ starts at 1), the path γ is larger than the path $\delta(t) = e^{tw(b)}$, which has length $|w(b)|$.

2.1. Locally exponential Lie groups

We will assume that the exponential map of K is a local diffeomorphism for *some open ball of the given Finsler semi-norm* μ. Adapting the technique from Riemannian geometry we obtain local minimality of segments.

Definition 11. Let K be a Lie group, $\mu = |\cdot|$ a bi-invariant Finsler semi-norm in $\mathrm{Lie}\,(K)$, and for $r > 0$ let $B\ \ = \{v \quad \mathrm{Lie}\,(K) : |v| < r\}$, $V_R = \exp(B_R)$. We assume in this section that *there exists* $R > 0$ *such that* V_R *is* τ_K *open in* K, *and* $\exp : B_R \to V_R$ *is a diffeomorphism.*

Remark 12. The straightforward example of our definition is given by a Banach–Lie group K and a bi-invariant norm $\mu = |\cdot|$ that is equivalent to the original norm modeling the Banach space $\mathrm{Lie}\,(K)$. Then the radius $R > 0$ is given by the fact that exp is a local diffeomorphism.

Theorem 13 (Local minimality of segments). *Let* $u_0, u_1 = u_0 e^z \quad K$ *with* $|z| < R$.

1. *Let* γ *be a piecewise* C^1 *path joining* u_0 *and* u_1. *If* γ *leaves* $u_0 V_R$, *then* $\mathrm{Length}_\mu(\gamma) \geq R$.
2. *If* $\delta(t) = u_0 e^{tz}$, $t \quad [0,1]$, *then* δ *is shorter that any other piecewise* C^1 *path* γ *in* K *joining* u_0, u_1 *and* $\mathrm{dist}_\mu(u_0, u_1) = |z|$.

By a standard approximation argument, we get

Corollary 14. *If* $u_1 = u_0 e^z$ *with* $|z| \leq R$, *the path* $\delta(t) = u_0 e^{tz}$ *is shorter than any other piecewise smooth path in* K *joining them.*

Corollary 15. *If* $u_0, u \quad K$ *and* $\mathrm{dist}_\mu(u_0, u_1) < R$, *then there exists* $z \quad \mathrm{Lie}\,(K)$ *with* $|z| < R$ *so that* $u = u_0 e^z$, *therefore* $|z| = \mathrm{dist}_\mu(u_0, u) = \mathrm{Length}_\mu(\delta)$, *where* δ *is the segment generated by* z. *In particular,* $\exp(B\ \ ) = V\ \ = \{u \quad K : \mathrm{dist}_\mu(u, 1) < r\}$ *holds for all* $r \leq R$.

2.1.1. Uniqueness and the EMD property. For strictly convex Finsler norms we now establish the uniqueness of segments as minimizing paths when the metric is strictly convex.

Theorem 16 (Uniqueness for strictly convex norms). *Assume that* $\mu = |\cdot|$ *is strictly convex. Let* $u_0, u_1 \quad K$ *and* $\gamma : [a,b] \to K$ *be a short rectifiable path joining* u_0, u_1. *Then there exists* $z \quad \mathrm{Lie}\,(K)$ *such that* $u_1 = u_0 e^z$ *and* γ *is a reparameterization of the segment* $\delta(t) = u_0 e^{tz}$. *If* $\mathrm{dist}_\mu(u_0, u_1) < R$, *then this segment is unique.*

Remark 17. The previous theorem shows than in the case of strictly convex Finsler norms, there exists a short path γ joining $1, u$ in K only if $u = e^z$ is in the range of the exponential map, and γ is then a segment. Moreover, if u is close to 1, this segment is unique. It is unclear however what is the maximal neighborhood of $0 \quad \mathrm{Lie}\,(K)$ where the segments are short paths, though one would expect that this would be the case when the exponential map is a diffeomorphism along the segment (a set which can be much larger than the μ-ball of radius R).

We now compare the distance in the manifold with the tangent distance and give criteria for equality to hold.

Theorem 18 (The Exponential Metric Decreasing (EMD) property). *If* v, w $\mathrm{Lie}\,(K)$ *then* $\mathrm{dist}_\mu(e^v, e^w) \leq |w - v|$.

1. *If* w, v *commute and* $|w - v| \leq R$*, then equality holds.*
2. *If equality holds and the norm is strictly convex, then* w, v *commute.*

Remark 19. For manifolds of non-positive curvature, one obtains a reversed inequality known as the EMI (exponential metric increasing property). See [9, Sections 3 and 4.1.4] and the references therein. Therefore our EMD speaks of the non-negative nature of the curvature of bi-invariant metrics on Lie groups.

At the Lie algebra level, the EMD property indicates a contraction property for the local Lie group structure product:

Corollary 20 (The BCH contraction property). *If* v, w $\mathrm{Lie}\,(K)$ *are such that* $e^v e^w$ V_R *(for instance, if* $|v + w| < R$*), then* $|v \star w| = \left|\exp^{-1}(e^v e^w)\right| \leq |v + w|$.

Proof. The hypothesis tells us that $z = \exp^{-1}(e^v e^w)$ is in B_R therefore by the minimality of segments and the EMD property $|z| = \mathrm{dist}_\mu(1, e^z) = \mathrm{dist}_\mu(e^v, e^{-w}) \leq |v + w|$. □

Unit spheres with faces. We will now establish some facts for semi-norms that are not strictly convex.

The following two corollaries tells us that, at least locally, for $\gamma(1) = e^z$ with z not lying in the intersection of maximal faces of the sphere of the normed space E_μ, a short path γ in K is lifted to an *also short path* Γ in $\mathrm{Lie}\,(K) \simeq E_\mu$ with the same length, thus characterizing these last paths suffices to characterize γ. This is related to the results obtained by Bialy and Polterovich et al. [6,11,12] for the Hofer metric ingroups of symplectomorphisms. However, we remark that the Weinstein chart is what is used in that context to prove that symplectomorphisms are *locally flat* for the Hofer metric.

Corollary 21. *Let* $\gamma : [a, b] \to K$ *be a short rectifiable path joining* 1 *and* $u = e^z$ *with* $|z| < R$ *and* $z\,|z|^{-1}$ *in the interior a maximal face of the unit sphere of* E_μ*. Then* $\gamma = e^\Gamma$ *for a short rectifiable path* Γ *joining* $0, z$ *in* E_μ *with the same length. Moreover, if* γ *is regular, then after normalizing, the speed* $\gamma_t^{-1}\dot{\gamma}$ *stays inside the same maximal face than* z*, for all* t $[a, b]$*.*

Assume that φ is a unit norm functional such that $\varphi(w - v) = |w - v|$ and φ vanishes in each other term of the BCH formula of $w, -v$. Then

$$\begin{aligned} 0 &\geq |w - v| - \mathrm{dist}_\mu(e^v, e^w) = |w - v| - |BCH(w, -v)| \\ &\geq \varphi(w - v) - \varphi(w - v + z) = 0, \end{aligned}$$

thus $\mathrm{dist}_\mu(e^v, e^w) = |w - v|$ and the EMD inequality turns into inequality. Using the previous results we can prove:

Corollary 22. *If* $v, w \quad \mathrm{Lie}\,(K)$ *are such that* $\mathrm{dist}_\mu(e^v, e^w) = |w - v| < R$*, and* $e^w e^{-v} = e^z$ *with* z *in a maximal face of the sphere, then there exists a unit norm functional such that* $\varphi(w - v) = |w - v|$ *and* $\varphi = 0$ *in every term of the the BCH expansion of* $e^v e^{-w}$*.*

§ Is the above condition equivalent to the assertion: φ gives the norm of $w - v$, and φ vanishes on the *Lie ideal generated* by v, w? That is, φ vanishes on each $z = [v, x] + [w, y]$ for x, y in the Lie algebra generated by v, w?

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Part V

Miscellaneous

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On Conjugacy of Subalgebras of Graph C^*-Algebras

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Abstract. The problem of inner vs outer conjugacy of subalgebras of certain graph C^*-algebras is investigated. For a large class of finite graphs E, we show that whenever α is a vertex-fixing quasi-free automorphism of the corresponding graph C^*-algebra $C^*(E)$ such that $\alpha(\mathcal{D}_E) \neq \mathcal{D}_E$, where $\mathcal{D}_E$ is the canonical MASA in $C^*(E)$, then $\alpha(\mathcal{D}_E) \neq w\mathcal{D}_E w^*$ for all unitaries $w \in C^*(E)$. That is, the two MASAs $\mathcal{D}_E$ and $\alpha(\mathcal{D}_E)$ of $C^*(E)$ are outer but not inner conjugate. For the Cuntz algebras $\mathcal{O}_n$, we find a criterion which guarantees that a polynomial automorphism moves the canonical UHF subalgebra to a non-inner conjugate UHF subalgebra. The criterion is phrased in terms of rescaling of trace on diagonal projections.

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Keywords. graph C^*-algebra; Cuntz algebra; MASA; UHF-subalgebra; automorphism, inner and outer conjugacy.

1. Introduction

Maximal abelian subalgebras (MASAs) have played very important role in the study of von Neumann algebras from the very beginning, and their theory is quite well developed by now. Theory of MASAs of C^*-algebras is somewhat less advanced, several nice attempts in this direction notwithstanding. Our particular interest lies in classification of MASAs in purely infinite simple C^*-algebras, and

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especially in Kirchberg algebras. In addition to its intrinsic interest, better understanding of MASAs in Kirchberg algebras could have significant consequences for the classification of automorphisms and group actions on these algebras. In this context, we would like to single out the recent work of Barlak and Li [2], where a connection between the outstanding UCT problem for crossed products and existence of invariant Cartan subalgebras is investigated.

It is a very difficult problem if two outer conjugate MASAs (that is, two MASAs $\mathfrak{A}$ and $\mathfrak{B}$ for which there exists an automorphism σ of the ambient algebra such that $\sigma(\mathfrak{A}) = \mathfrak{B}$) of a purely infinite simple C^*-algebra are inner conjugate as well (that is, if there exists a unitary w such that $w\mathfrak{A}w^* = \mathfrak{B}$). This question was answered to the negative in [7, Theorem 3.7] for quasi-free automorphisms of the Cuntz algebras $\mathcal{O}_n$.

In the present paper, we extend the main result of [7] to the case of purely infinite simple graph C^*-algebras $C^*(E)$ corresponding to finite graphs E. Namely, we show in Theorem 2 below that every quasi-free automorphism of $C^*(E)$ either leaves the canonical MASA $\mathcal{D}_E$ globally invariant or moves it to another MASA of $C^*(E)$ which is not inner conjugate to $\mathcal{D}_E$. Although our Theorem 2 is stated for quasi-free automorphisms only, it is in fact applicable to some other automorphisms as well. This is due to the fact that passing from one graph E to another F with the isomorphic algebra $C^*(F) \cong C^*(E)$ will often not preserve the property of an automorphism to be quasi-free. To make the present paper self-contained, we recall the necessary background on graph C^*-algebras and their endomorphisms in the preliminaries.

The problem of conjugacy of subalgebras has been mostly investigated in the context of MASAs. However, it is very interesting for other types of subalgebras as well. In the present paper, we initiate systematic investigations of the outer vs inner conjugacy for the canonical UHF-subalgebra $\mathcal{F}_n$ of the Cuntz algebra $\mathcal{O}_n$. More specifically, we address the question if $\mathcal{F}_n$ may be inner conjugate to $\lambda_u(\mathcal{F}_n)$, where λ_u is a polynomial automorphism of $\mathcal{O}_n$, building on the first observations in this direction made in [5]. Our results have clear potential for shedding more light on the mysterious structure of the outer automorphism group of $\mathcal{O}_n$.

2. Preliminaries

2.1. Finite directed graphs and their C^*-algebras

Let $E = (E^0, E^1, r, s)$ be a directed graph, where E^0 and E^1 are *finite* sets of vertices and edges, respectively, and $r, s : E^1 \to E^0$ are range and source maps, respectively. A *path* μ of length $|\mu| = k \geq 1$ is a sequence $\mu = (\mu_1, \dots, \mu_k)$ of k edges μ_j such that $r(\mu_j) = s(\mu_{j+1})$ for $j = 1, \dots, k-1$. We view the vertices as paths of length 0. The set of all paths of length k is denoted E^k, and E^* denotes the collection of all finite paths (including paths of length zero). The range and source maps naturally extend from edges E^1 to paths E^k. A *sink* is a vertex v which emits no edges, i.e. $s^{-1}(v) = \emptyset$. By a *cycle* we mean a path μ of length

$|\mu| \geq 1$ such that $s(\mu) = r(\mu)$. A cycle $\mu = (\mu_1, \ldots, \mu_k)$ has an exit if there is a j such that $s(\mu_j)$ emits at least two distinct edges. Graph E is *transitive* if for any two vertices v, w there exists a path $\mu \in E^*$ from v to w of non-zero length. Thus a transitive graph does not contain any sinks or sources. Given a graph E, we will denote by $A = [A(v,w)]_{v,w\in E^0}$ its *adjacency matrix*. That is, A is a matrix with rows and columns indexed by the vertices of E, such that $A(v,w)$ is the number of edges with source v and range w.

The C^*-algebra $C^*(E)$ corresponding to a graph E is by definition, [16] and [15], the universal C^*-algebra generated by mutually orthogonal projections P_v, $v \in E^0$, and partial isometries S_e, $e \in E^1$, subject to the following two relations:

(GA1) $S_e^* S_e = P_{r(e)}$,
(GA2) $P_v = \sum_{s(e)=v} S_e S_e^*$ if $v \in E^0$ emits at least one edge.

For a path $\mu = (\mu_1, \ldots, \mu_k)$ we denote by $S_\mu = S_{\mu_1} \cdots S_{\mu_k}$ the corresponding partial isometry in $C^*(E)$. We agree to write $S_v = P_v$ for a $v \in E^0$. Each S_μ is non-zero with the domain projection $P_{r(\mu)}$. Then $C^*(E)$ is the closed span of $\{S_\mu S_\nu^* : \mu, \nu \in E^*\}$. Note that $S_\mu S_\nu^*$ is non-zero if and only if $r(\mu) = r(\nu)$. In that case, $S_\mu S_\nu^*$ is a partial isometry with domain and range projections equal to $S_\nu S_\nu^*$ and $S_\mu S_\mu^*$, respectively.

The range projections $P_\mu = S_\mu S_\mu^*$ of all partial isometries S_μ mutually commute, and the abelian C^*-subalgebra of $C^*(E)$ generated by all of them is called the diagonal subalgebra and denoted $\mathcal{D}_E$. We set $\mathcal{D}_E^0 = \mathrm{span}\{P_v : v \in E^0\}$ and, more generally, $\mathcal{D}_E^k = \mathrm{span}\{P_\mu : \mu \in E^k\}$ for $k \geq 0$. C^*-algebra $\mathcal{D}_E$ coincides with the closed linear span of $\bigcup_{k=0}^\infty \mathcal{D}_E^k$. If E does not contain sinks and all cycles have exits then $\mathcal{D}_E$ is a MASA (maximal abelian subalgebra) in $C^*(E)$ by [14, Theorem 5.2]. Throughout this paper, we make the following

Standing assumption: all graphs we consider are transitive and all cycles in these graphs admit exits.

There exists a strongly continuous action γ of the circle group $U(1)$ on $C^*(E)$, called the *gauge action*, such that $\gamma_z(S_e) = zS_e$ and $\gamma_z(P_v) = P_v$ for all $e \in E^1$, $v \in E^0$ and $z \in U(1) \subseteq \mathbb{C}$. The fixed-point algebra $C^*(E)^\gamma$ for the gauge action is an AF-algebra, denoted $\mathcal{F}_E$ and called the core AF-subalgebra of $C^*(E)$. $\mathcal{F}_E$ is the closed span of $\{S_\mu S_\nu^* : \mu, \nu \in E^*,\ |\mu| = |\nu|\}$. For $k \in \mathbb{N} = \{0, 1, 2, \ldots\}$ we denote by $\mathcal{F}_E^k$ the linear span of $\{S_\mu S_\nu^* : \mu, \nu \in E^*,\ |\mu| = |\nu| = k\}$. C^*-algebra $\mathcal{F}_E$ coincides with the norm closure of $\bigcup_{k=0}^\infty \mathcal{F}_E^k$.

We consider the usual *shift* on $C^*(E)$, [13], given by

$$\varphi(x) = \sum_{e\in E^1} S_e x S_e^*, \quad x \in C^*(E). \tag{1}$$

In general, for finite graphs without sinks and sources, the shift is a unital, completely positive map. However, it is an injective $*$-homomorphism when restricted to the relative commutant $(\mathcal{D}_E^0)' \cap C^*(E)$.

We observe that for each $v \in E^0$ projection $\varphi^k(P_v)$ is minimal in the center of $\mathcal{F}_E^k$. The C^*-algebra $\mathcal{F}_E^k\varphi^k(P_v)$ is the linear span of partial isometries $S_\mu S_\nu^*$ with $|\mu| = |\nu| = k$ and $r(\mu) = r(\nu) = v$. It is isomorphic to the full matrix algebra of size $\sum_{w\in E^0} A^k(w,v)$. The multiplicity of $\mathcal{F}_E^k\varphi^k(P_v)$ in $\mathcal{F}_E^{k+1}\varphi^{k+1}(P_w)$ is $A(v,w)$, so the Bratteli diagram for $\mathcal{F}_E$ is induced from the graph E, see [13], [16] or [3]. We also note that the relative commutant of $\mathcal{F}_E^k$ in $\mathcal{F}_E^{k+1}$ is isomorphic to $\bigoplus_{v,w\in E^0} M_{A(v,w)}(\mathbb{C})$.

For an integer $m \in \mathbb{Z}$, we denote by $C^*(E)^{(m)}$ the spectral subspace of the gauge action corresponding to m. That is,

$$C^*(E)^{(m)} := \{x \in C^*(E) \mid \gamma_z(x) = z^m x, \forall z \in U(1)\}.$$

In particular, $C^*(E)^{(0)} = C^*(E)^\gamma$. There exist faithful conditional expectations $\Phi_{\mathcal{F}} : C^*(E) \to \mathcal{F}_E$ and $\Phi_{\mathcal{D}} : C^*(E) \to \mathcal{D}_E$ such that $\Phi_{\mathcal{F}}(S_\mu S_\nu^*) = 0$ for $|\mu| \neq |\nu|$ and $\Phi_{\mathcal{D}}(S_\mu S_\nu^*) = 0$ for $\mu \neq \nu$. Combining $\Phi_{\mathcal{F}}$ with a faithful conditional expectation from $\mathcal{F}_E$ onto $\mathcal{F}_E^k$, we obtain a faithful conditional expectation $\Phi_{\mathcal{F}}^k : C^*(E) \to \mathcal{F}_E^k$. Furthermore, for each $m \in \mathbb{Z}$ there is a unital, contractive and completely bounded map $\Phi^m : C^*(E) \to C^*(E)^{(m)}$ given by

$$\Phi^m(x) = \int_{z\in U(1)} z^{-m}\gamma_z(x)dz. \tag{2}$$

In particular, $\Phi^0 = \Phi_{\mathcal{F}}$. We have $\Phi^m(x) = x$ for all $x \in C^*(E)^{(m)}$. If $x \in C^*(E)$ and $\Phi^m(x) = 0$ for all $m \in \mathbb{Z}$ then $x = 0$.

2.2. Endomorphisms determined by unitaries

Cuntz's classical approach to the study of endomorphisms of $\mathcal{O}_n$, [12], has recently been extended to graph C^*-algebras in [7] and [1]. In this subsection, we recall a few most essential definitions and facts about such endomorphisms.

We denote by $\mathcal{U}_E$ the collection of all those unitaries in $C^*(E)$ which commute with all vertex projections P_v, $v \in E^0$. That is

$$\mathcal{U}_E := \mathcal{U}\big((\mathcal{D}_E^0)' \cap C^*(E)\big). \tag{3}$$

If $u \in \mathcal{U}_E$ then uS_e, $e \in E^1$, are partial isometries in $C^*(E)$ which together with projections P_v, $v \in E^0$, satisfy (GA1) and (GA2). Thus, by the universality of $C^*(E)$, there exists a unital $*$-homomorphism $\lambda_u : C^*(E) \to C^*(E)$ such that[1]

$$\lambda_u(S_e) = uS_e \quad \text{and} \quad \lambda_u(P_v) = P_v, \quad \text{for } e \in E^1,\ v \in E^0. \tag{4}$$

The mapping $u \mapsto \lambda_u$ establishes a bijective correspondence between $\mathcal{U}_E$ and the semigroup of those unital endomomorphisms of $C^*(E)$ which fix all P_v, $v \in E^0$. As observed in [6, Proposition 2.1], if $u \in \mathcal{U}_E \cap \mathcal{F}_E$ then λ_u is automatically injective. We say λ_u is *invertible* if λ_u is an automorphism of $C^*(E)$. We denote

$$\mathfrak{B} := (\mathcal{D}_E^0)' \cap \mathcal{F}_E^1. \tag{5}$$

[1] The reader should be aware that in some papers (e.g. in [12]) a different convention is used, namely $\lambda_u(S_e) = u^*S_e$.

That is, $\mathfrak{B}$ is the linear span of elements $S_eS_f^*$, $e,f \in E^1$, with $s(e)=s(f)$ and $r(e)=r(f)$. We note that $\mathfrak{B}$ is contained in the multiplicative domain of φ and we have $\mathcal{D}_E^1 \subseteq \mathfrak{B} \subseteq \mathcal{F}_E^1$. If $u \in \mathcal{U}(\mathfrak{B})$ then λ_u is automatically invertible with inverse λ_{u^*} and the map

$$\mathcal{U}(\mathfrak{B}) \ni u \mapsto \lambda_u \in \mathrm{Aut}\,(C^*(E)) \tag{6}$$

is a group homomorphism with range inside the subgroup of *quasi-free automorphisms* of $C^*(E)$, see [17]. Note that this group is almost never trivial and it is non-commutative if graph E contains two edges $e,f \in E^1$ such that $s(e)=s(f)$ and $r(e)=r(f)$.

The shift φ globally preserves $\mathcal{U}_E$, $\mathcal{F}_E$ and $\mathcal{D}_E$. For $k \geq 1$ we denote

$$u_k := u\varphi(u)\cdots\varphi^{k-1}(u). \tag{7}$$

For each $u \in \mathcal{U}_E$ and all $e \in E^1$ we have $S_eu = \varphi(u)S_e$, and thus

$$\lambda_u(S_\mu S_\nu^*) = u_{|\mu|}S_\mu S_\nu^* u_{|\nu|}^* \tag{8}$$

for any two paths $\mu,\nu \in E^*$.

3. Quasi-free automorphisms

In this section, we extend the main result of [7], applicable to the Cuntz algebras, to a much wider class of graph C^*-algebras.

For the proof of Lemma 1, below, we recall from Lemma 3.2 and Remark 3.3 in [7] that if $x \in C^*(E)$, $x \geq 0$, and $x\mathcal{D}_E = \mathcal{D}_E x$ then $x \in \mathcal{D}_E$.

Lemma 1. *Let $u \in \mathcal{U}(\mathfrak{B})$ be such that $u\mathcal{D}_E^1u^* \neq \mathcal{D}_E^1$, and let $x \in \mathcal{F}_E$ be arbitrary. If $x\lambda_u(\mathcal{D}_E) = \mathcal{D}_E x$ then $x=0$.*

Proof. Suppose $x \in \mathcal{F}_E$ is such that $\|x\|=1$ and $x\lambda_u(\mathcal{D}_E) = \mathcal{D}_E x$. From this we will derive a contradiction.

Since $u\mathcal{D}_E^1u^* \neq \mathcal{D}_E^1$, there exists a vertex $v \in E^0$ such that $u\mathcal{D}_E^1u^*P_v \neq \mathcal{D}_E^1P_v$. Thus, since $u\mathcal{D}_E^1P_vu^* = u\mathcal{D}_E^1u^*P_v$, we can take a projection $p \in \mathcal{D}_E^1P_v$ satisfying $\delta := \inf\left\{\|upu^*-q\| \mid q \in \mathcal{D}_E^1\right\} > 0$. Since $\Phi_{\mathcal{F}}^1(q') \in \mathcal{D}_E^1$, for all $q' \in \mathcal{D}_E$ we get

$$\|upu^*-q'\| \geq \|\Phi_{\mathcal{F}}^1(upu^*-q')\| = \|upu^*-\Phi_{\mathcal{F}}^1(q')\| \geq \delta.$$

By assumption, for each $k \in \mathbb{N}$ there is a $q_k \in \mathcal{D}_E$ such that

$$x\lambda_u(\varphi^k(p)) = q_kx. \tag{9}$$

Since $u_k \in \mathcal{F}_E^k$ and $\varphi^k(upu^*) \in \varphi^k(\mathfrak{B}) = (\mathcal{F}_E^k)' \cap \mathcal{F}_E^{k+1}$, we have

$$\lambda_u(\varphi^k(p)) = u_k\varphi^k(\lambda_u(p))u_k^* = u_k\varphi^k(upu^*)u_k^* = \varphi^k(upu^*). \tag{10}$$

Identities (9) and (10) combined yield $0 = x\lambda_u(\varphi^k(p)) - q_kx = x\varphi^k(upu^*) - q_kx$. Since $upu^* \in \mathfrak{B}$, the sequence $\left\{\varphi^k(upu^*)\right\}_{k=1}^\infty$ is central in $\mathcal{F}_E$. Therefore we have $\lim_{k\to\infty}(\varphi^k(upu^*)-q_k)xx^* = 0$. It follows from the assumption on x that $xx^*\mathcal{D}_E = \mathcal{D}_E xx^*$, and thus we may conclude that $xx^* \in \mathcal{D}_E$.

Now, take an arbitrary $0 < \epsilon < 1$. For a sufficiently large $m \quad \mathbb{N}$, we have

$$\limsup_{k\to\infty} \left\|(\varphi^k(upu^*) - q_k)\Phi^m_{\mathcal{F}}(xx^*)\right\| \le \epsilon \quad \text{and} \quad \left\|\Phi^m_{\mathcal{F}}(xx^*)\right\| \ge 1-\epsilon.$$

Thus we can find a projection $d \quad \mathcal{D}^m_E$ such that

$$\limsup_{k\to\infty} \left\|(\varphi^k(upu^*) - q_k)d\right\| \le \frac{\epsilon}{1-\epsilon}.$$

Since graph E is transitive, for a sufficiently large $k \quad \mathbb{N}$ we can find a path $\mu \quad E^k$ such that $r(\mu) = v$ and $S_\mu S_\mu^* \le d$. But now we see that

$$3\epsilon \ge \left\|(\varphi^k(upu^*) - q_k)d\right\| \ge \left\|(\varphi^k(upu^*) - q_k)S_\mu S_\mu^*\right\| = \left\|upu^*P_v - S_\mu^* q_k S_\mu\right\| \ge \delta.$$

Since ϵ can be arbitrarily small, this is the desired contradiction. □

Now, we are ready to prove our main result.

Theorem 2. *Let $u \quad \mathcal{U}(\mathfrak{B})$ be such that $u\mathcal{D}^1_E u^* \ne \mathcal{D}^1_E$. Then there is no non-zero element $x \quad C^*(E)$ satisfying $x\lambda_u(\mathcal{D}_E) = \mathcal{D}_E x$. In particular, there is no unitary $w \quad C^*(E)$ such that $w\mathcal{D}_E w^* = \lambda_u(\mathcal{D}_E)$.*

Proof. Let $x \quad C^*(E)$ be such that $x\lambda_u(\mathcal{D}_E) = \mathcal{D}_E x$. To verify that $x = 0$, it suffices to show that $\Phi^m(x) = 0$ for all $m \quad \mathbb{Z}$.

We have $S_\mu^* \mathcal{D}_E S_\mu = P_{\ (\mu)}\mathcal{D}_E$ for each $\mu \quad E^*$. Thus $P_{\ (\mu)} x\lambda_u(\mathcal{D}_E) = P_{\ (\mu)}\mathcal{D}_E x = S_\mu^*\mathcal{D}_E S_\mu x$, and hence $S_\mu x\lambda_u(\mathcal{D}_E) = \mathcal{D}_E S_\mu x$. Therefore by Lemma 1, we get

$$\Phi_{\mathcal{F}}(S_\mu x) = 0 \quad \text{for all} \quad \mu \quad E^*.$$

Let $m \quad \mathbb{N}$. For a vertex $v \quad E^0$ take a path $\mu \quad E^m$ with $r(\mu) = v$. Then $0 = \Phi_{\mathcal{F}}(S_\mu x) = S_\mu \Phi^{-m}(x)$. Thus $P_v\Phi^{-m}(x) = 0$, and summing over all $v \quad E^0$ we see that $\Phi^{-m}(x) = 0$ for all $m \quad \mathbb{N}$.

Now, taking adjoints of both sides of the identity $x\lambda_u(\mathcal{D}_E) = \mathcal{D}_E x$ and then applying $\lambda_{u^*} = \lambda_u^{-1}$, we get $\lambda_{u^*}(x^*)\lambda_{u^*}(\mathcal{D}_E) = \mathcal{D}_E\lambda_{u^*}(x^*)$. Since $u^* \quad \mathcal{U}(\mathfrak{B})$ and $u^*\mathcal{D}^1_E u \ne \mathcal{D}^1_E$, applying the preceding argument, we get $\Phi^{-m}\big(\lambda_{u^*}(x^*)\big) = 0$ for all $m \quad \mathbb{N}$. But $\Phi^{-m}\big(\lambda_{u^*}(x^*)\big) = \lambda_{u^*}\big(\Phi^{-m}(x^*)\big) = \lambda_{u^*}\big(\Phi^m(x)\big)$. Thus $\Phi^m(x) = 0$ for all $m \quad \mathbb{N}$, and the proof is complete. □

Corollary 3. *Let $u, v \quad \mathcal{U}(\mathfrak{B})$ be such that $u\mathcal{D}^1_E u^* \ne v\mathcal{D}^1_E v^*$. Then there is no unitary $w \quad C^*(E)$ such that $w\lambda_u(\mathcal{D}_E)w^* = \lambda_v(\mathcal{D}_E)$.*

4. Conjugacy by polynomial automorphisms of the UHF-subalgebra of the Cuntz algebra

In this section we give a condition for inner conjugacy by polynomial automorphism of the core UHF-subalgebra of the Cuntz algebra $\mathcal{O}_n$, using the unique normalized trace on $\mathcal{F}_n$ which will be denoted by τ.

Let W_n^k be the set of tuples $\mu = (\mu_1, \ldots, \mu_k)$ where $\mu_i \quad \{1, \ldots, n\}$, and define $W_n = \bigcup_{k=0}^\infty W_n^k$ where $W_n^0 = \{0\}$. We denote by $\mathcal{S}_n$ the group of unitaries in $\mathcal{O}_n$ which can be written as finite sums of words. Hence an element $u \quad \mathcal{S}_n$ is of the form $u = \sum_{(\alpha,\beta)\in J} S_\alpha S_\beta^*$, where J is a finite collection of pairs (α, β) with $\alpha, \beta \quad W_n$. In the following we denote by $P(\mathcal{D}_n)$ the set of projections in $\mathcal{D}_n$.

Theorem 4. *Let* $u \quad \mathcal{S}_n$ *with* $\lambda_u \quad \mathrm{Aut}\,(\mathcal{O}_n)$. *If there exists a sequence* $\{P_n\}$ *in* $P(\mathcal{D}_n)$ *such that*

$$\frac{\tau\big(\lambda_u(P_n)\big)}{\tau(P_n)} \to \infty \text{ or } 0, \tag{11}$$

then for all $w \quad \mathcal{U}(\mathcal{O}_n)$ *and* $v \quad \mathcal{U}(\mathcal{F}_n)$ *we have* $\lambda_u \neq \mathrm{Ad}\, w\lambda_v$. *This implies in particular that* $\mathcal{F}_n$ *and* $\lambda_u(\mathcal{F}_n)$ *are not inner conjugate.*

For the proof of Theorem 4 we need the following result.

Lemma 5. *If* $u \quad \mathcal{U}(\mathcal{O}_n)$ *such that* $u\mathcal{D}_n u^* \subseteq \mathcal{F}_n$ *then* u *has a finite Fourier series.*

Proof. Let u have the Fourier series $\sum_{k\in\mathbb{Z}} u_k,\ u_k := \Phi^k(u)$. Let $P \quad \mathcal{D}_n$ and fix $k \quad \mathbb{Z}$ then

$$uPu^*u_k = \int_{U(1)} z^{-k}\gamma_z(uPu^*u)dz = \int_{U(1)} z^{-k}\gamma_z(uP)dt = u_kP.$$

Here we have used that $u\mathcal{D}_n u^* \subseteq \mathcal{F}_n$ to write $uPu^*\gamma_z(u) = \gamma_z(uPu^*u)$.

It follows that $P(u^*u_k) = (u^*u_k)P$ for $P \quad \mathcal{D}_n$. Hence $u^*u_k \quad \mathcal{D}_n' \cap \mathcal{O}_n$ and since $\mathcal{D}_n$ is a MASA there exists a $d_k \quad \mathcal{D}_n$ such that $u_k = ud_k$. Now we have

$$u_k = \int_{U(1)} z^{-k}\gamma_z(u_k)dz = \int_{U(1)} z^{-k}\gamma_z(ud_k)dz = u_kd_k.$$

Hence $ud_k = ud_kd_k$ and $d_k = d_k^2$, which shows that d_k is a projection in $\mathcal{D}_n$. For $k, m \quad \mathbb{Z}$ with $k \neq m$ we have

$$0 = \int_{U(1)} \gamma_z(u_k^*u_m)dz = \int_{U(1)} \gamma_z(d_k^*d_m)dz = d_kd_m.$$

Hence $\{d_k\}$ are mutually orthogonal projections in $\mathcal{D}_n$.

There exists an element of the form $w = \sum_j t_jS_{\alpha_j}S_{\beta_j}^*$ (finite sum), $t_j \quad \mathbb{C}$, such that $\|u - w\| < \epsilon$. Also, there exists $N \quad \mathbb{Z}$ such that for $k > N$ we have $\Phi^k(w) = 0$, since w has a finite Fourier series. Then

$$\|u_k\| = \|\Phi^k(u)\| = \|\Phi^k(u - w)\| \leq \|u - w\| < \epsilon, \text{ for } k > N.$$

Hence $\|u_k\| \to 0$ and thus $\|d_k\| \to 0$. Since d_k are all projections the series $\{d_k\}$ is finite, concluding that u has a finite Fourier series since $u_k = ud_k$. $\square$

Proof of Theorem 4. We recall that if $\alpha \quad \mathrm{Aut}\,(\mathcal{O}_n)$ then $\mathcal{F}_n$ and $\alpha(\mathcal{F}_n)$ are inner conjugate if and only if there exists $w \quad \mathcal{U}(\mathcal{O}_n)$ such that $\alpha(\mathcal{F}_n) = \mathrm{Ad}\, w(\mathcal{F}_n)$. This is equivalent to that $\mathrm{Ad}\,(w^*)\,\alpha(\mathcal{F}_n) = \mathcal{F}_n$. If we have an automorphism which globally preserves $\mathcal{F}_n$ then $\mathrm{Ad}\,(w^*)\alpha$ is equal λ_v for some $v \quad \mathcal{U}(\mathcal{F}_n)$ [12, Proposition 1.2(b)], [10, Proposition 3.3] therefore $\alpha = \mathrm{Ad}\, w\lambda_v$. Hence $\mathcal{F}_n$ and $\alpha(F_n)$ are inner conjugate if and only if there exist $w \quad \mathcal{U}(\mathcal{O}_n)$ and $v \quad \mathcal{U}(\mathcal{F}_n)$ s.t. $\alpha = \mathrm{Ad}\, w\lambda_v$.

Assume that $\lambda_u = \operatorname{Ad} w \lambda_v$ for $w \quad \mathcal{U}(\mathcal{O}_n)$ and $v \quad \mathcal{U}(\mathcal{F}_n)$. We will assume $w \ / \ \mathcal{F}_n$ otherwise $\operatorname{Ad} w$ is trace preserving which contradicts (11) since λ_v is also trace preserving. We wish to prove that there are no sequences of projections such that we get (11).

Since $\lambda_u \quad \operatorname{Aut}(\mathcal{O}_n)$ we have $\lambda_v \quad \operatorname{Aut}(\mathcal{O}_n)$ and $\operatorname{Ad} w^* = \lambda_v \lambda_u^{-1}$. Therefore $\operatorname{Ad} w^*(\mathcal{D}_n) \subseteq \mathcal{F}_n$ [10, Proposition 3.3]. We wish to show that there exists $M > 0$ such that

$$\frac{\tau(w^* P w)}{\tau(P)} \le M, \text{ for } P \quad P(\mathcal{D}_n). \tag{12}$$

By Lemma 5, w^* has a finite Fourier series and we can write

$$w^* = \sum_{j=1}^{k} S_v^{*j} x_{-j} + x_0 + \sum_{j=1}^{k} x_j S_v^j,$$

with $v \quad \{1, 2, \ldots, n\}$ and $x_0, x_{\pm j} \quad \mathcal{F}_n$.

For any word μ with $|\mu| > k$ and $1 \le j \le k$, we can write $\mu = \nu_j \mu_j$ such that $|\nu_j| = j$. Then

$$\tau(P_\mu) = \frac{1}{n^{|\mu|}} = \frac{1}{n^j} \tau(P_{\mu_j}) \ge \frac{1}{n^k} \tau(P_{\mu_j}).$$

The only parts contributing to $\tau(w^* P w)$ are

$$\sum_{j=1}^{k} S_v^{*j} x_{-j} P x_{-j}^* S_v^j, \quad \sum_{j=1}^{k} x_j S_v^j P S_v^{*j} x_j^*, \quad x_0 P x_0^*.$$

We have $S_v^{*j} x_{-j} S_{\nu_j} \quad \mathcal{F}_n$ because $|v_j| = j$. Hence

$$\tau\big(S_v^{*j} x_{-j} P_\mu x_{-j}^* S_v^j\big) = \tau\big(\big(S_{\nu_j}^* x_{-j}^* S_v^j S_v^{*j} x_{-j} S_{\nu_j}\big) P_{\mu_j}\big).$$

Note that $S_v^{j*} x_{-j}$ is contractive since $\Phi^{-j}(w^*) = S_v^{j*} x_{-j}$ and Φ^j is contractive. By the Cauchy–Schwarz inequality, we get $\tau\big(S_v^{*j} x_{-j} P_\mu x_{-j}^* S_v^j\big) \le \tau\big(P_{\mu_j}\big) \le n^k \tau\big(P_\mu\big)$. On the other hand, using that $\Phi^j(w^*) = x_j S_v^j$ is contractive and the Cauchy–Schwarz inequality we have

$$\begin{aligned}\tau\big(x_j S_v^j P_\mu S_v^{*j} x_j^*\big) &= \tau\big(\big(x_j S_v^j S_v^{j*}\big)\big(S_v^j P_\mu S_v^{*j}\big)\big(S_v^j S_v^{j*} x_j^*\big)\big) \\ &\le \tau\big(S_v^j P_\mu S_v^{*j}\big) \le \tau\big(P_\mu\big).\end{aligned}$$

We also have $\tau\big(x_0^* P_\mu x_0\big) = \tau\big(x_0 x_0^* P_\mu\big) \le \tau\big(P_\mu\big)$ since $\Phi^0(w^*) = x_0$ which is contractive. Hence there exists a constant $M > 0$ satisfying $\tau\big(w^* P_\mu w\big) \le M \tau\big(P_\mu\big)$ for any word μ with $|\mu| > k$. If $|\mu| \le k$ we can extend the length until it is greater than k using that $\sum_{i=1}^n S_i S_i^* = 1$. Hence $\tau(w^* P w) \le M \tau(P)$ for any $P \quad P(\mathcal{D}_n)$. Since λ_v is trace preserving we have

$$\frac{\tau\big(\operatorname{Ad} w^* \lambda_u(P)\big)}{\tau(P)} = \frac{\tau\big(\lambda_v(P)\big)}{\tau(P)} = 1, \text{ for } P \quad P(\mathcal{D}_n). \tag{13}$$

Hence

$$\frac{1}{M} \le \frac{\tau\big(\lambda_u(P)\big)}{\tau(P)} \quad \text{for } P \quad P(\mathcal{D}_n)$$

since

$$\frac{\tau(P)}{\tau\big(\lambda_u(P)\big)} = \frac{\tau\big(\operatorname{Ad} w^*(\lambda_u(P))\big)}{\tau\big(\lambda_u(P)\big)} \leq M \quad \text{for } P \quad P(\mathcal{D}_n).$$

Then there are no sequence of projections $\{P_n\}$ such that $\dfrac{\tau\big(\lambda_u(P_n)\big)}{\tau(P_n)}$ tends to 0. To show that there is no sequence of projections such that the limit is infinite, we use that $\lambda_u(\mathcal{F}_n)$ is inner conjugate to $\mathcal{F}_n$ if and only if $\lambda_u^{-1}(\mathcal{F}_n)$ is inner conjugate to $\mathcal{F}_n$. Indeed, if $\lambda_u(\mathcal{F}_n) = w\mathcal{F}_n w^*$ then $\mathcal{F}_n = \lambda_u^{-1}(w)\lambda_u^{-1}(\mathcal{F}_n)\lambda_u^{-1}(w^*)$.

Now, assume that there exists a sequence $\{P_n\}$ such that $\dfrac{\tau\big(\lambda_u(P_n)\big)}{\tau(P_n)} \to \infty$, then $\dfrac{\tau(Q_n)}{\tau\big(\lambda_u^{-1}(Q_n)\big)} \to \infty$ where $Q_n = \lambda_u(P_n)$. But then $\{Q_n\}$ is a sequence of projections such that $\dfrac{\tau\big(\lambda_u^{-1}(Q_n)\big)}{\tau(Q_n)} \to 0$. Since for $u \quad \mathcal{S}_n$ we have $\lambda_u^{-1} = \lambda_v$ for some $v \quad \mathcal{S}_n$, [9, Theorem 2.1], we can use the same argument as above to show that there exists $M' > 0$ such that

$$\frac{1}{M'} \leq \frac{\tau\big(\lambda_u^{-1}(P)\big)}{\tau(P)} \quad \text{for } P \quad P(\mathcal{D}_n).$$

The claim follows. □

Example 6. Let $w = S_{22}S_{212}^* + S_{212}S_{22}^* + P_{211} + P_1 \quad \mathcal{S}_2$ then

$$\lambda_w(S_1) = S_1, \quad \lambda_w(S_2) = S_2(S_2S_{12}^* + S_{12}S_2^* + P_{11}).$$

Let $u = S_2S_{12}^* + S_{12}S_2^* + P_{11} \quad \mathcal{S}_2$, note that $u \quad \mathcal{U}\big(C^*(S_1)\big)$. Let α_u be defined by $\alpha_u(S_1) = S_1$ and $\alpha_u(S_2) = S_2u$, then α_u is an automorphism with inverse α_{u^*} and $\lambda_w = \alpha_u$. Hence λ_w is an automorphism of $\mathcal{O}_2$. Consider $\beta_k = (22\ldots 2)$, which is a word of length k only containing $2'$s and let $\gamma_k = (21212\ldots 12)$ be a word of length $2k-1$ then

$$\lambda_w(P_{\beta_k}) = S_2uS_2uS_2u\cdots S_2uu^*S_2^*u^*S_2^*\cdots u^*S_2^* = P_{\gamma_k}$$

since $uS_2 = S_{12}$. We then have

$$\frac{\tau\big(\lambda_w(P_{\beta_k})\big)}{\tau(P_{\lambda_k})} = \frac{1}{2^{k-1}}$$

which tends to 0 as $k \to \infty$. Hence $\mathcal{F}_2$ and $\lambda_w(\mathcal{F}_2)$ are not inner conjugate by Theorem 4 using the sequence $\big\{P_{\beta_k}\big\}$.

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A Direct Proof for an Eigenvalue Problem by Counting Lagrangian Submanifolds

Tomoyo Kanazawa

Abstract. We focus on one of the Schrödinger operators called the Bochner–Laplacian. Using Jensen's Formula and Vandermonde convolution, we show directly that for each $k = 0, 1, 2, \dots,$ the number of Lagrangian submanifolds which satisfy the Maslov quantization condition is just equal to the multiplicity of the kth eigenvalue of the operator.

Mathematics Subject Classification (2000). Primary 05A10; Secondary 53D12.

Keywords. Jensen's Formula, Vandermonde convolution, Bochner–Laplacian.

1. Introduction

In this paper, we prove precisely that the following two numerical expressions given by R. Kuwabara [4] and A. Yoshioka [1] are quite identical.

The Bochner–Laplacian is a Schrödinger operator which operates on the space of C^∞ sections of E_m, Hermitian line bundle over the complex projective space $\mathbb{C}P^n$. That is to say, on each line bundle E_m where m $\mathbb{Z}$ is called the Chern number, there is a unique harmonic connection denoted by $\tilde{d}_m$. Kuwabara computed the following spectrum of the Bochner–Laplacian associated with $\tilde{d}_m$ (see [4], 204p).

Proposition 1 ([4, Proposition 2.3]). *The eigenvalues of the Bochner–Laplacian associated with* $\tilde{d}_m$ *are*

$$\lambda_m^{(k)} = \left(k + \frac{|m|}{2}\right)\left(k + \frac{|m|}{2} + n\right) - \frac{m^2}{}, \quad k = 0, 1, 2, \dots$$

and the multiplicity of $\lambda_m^{(k)}$ *is*

$$\binom{k + |m| + n}{k + |m|}\binom{k + n}{k} - \binom{k + |m| + n - 1}{k + |m| - 1}\binom{k + n - 1}{k - 1}, \tag{1}$$

where $n \quad \mathbb{N}$ *is the dimension of the complex projective space* $\mathbb{C}P^n$.

On the other hand, Yoshioka also computed them in a different way called quasi-classical calculation of eigenvalues. From this viewpoint, in the cotangent bundle $T^*\mathbb{C}P^n$ called the phase space of $\mathbb{C}P^n$, they determined Lagrangian submanifolds which satisfy the Maslov quantization condition. Eigenvalues are invariants on each submanifold, and multiplicities are interpreted as the numbers of Lagrangian submanifolds which have the same eigenvalue. With this formulation, the quasi-classical eigenvalues of the Bochner–Laplacian are as follows (see [1, p. 55]).

Theorem 2 ([1, Theorem 4.4 & Lemma 4.3]). *The quasi-classical eigenvalues of the Bochner–Laplacian are*

$$\widetilde{\lambda}_m^{(k)} = \left(k + \frac{|m|}{2}\right)\left(k + \frac{|m|}{2} + n\right) - \frac{m^2}{\ } + \frac{n^2}{\ } = \lambda_m^{(k)} + \frac{n^2}{\ },$$

and for each $k = 0, 1, 2, \ldots,$ *the number of Lagrangian submanifolds satisfying the Maslov quantization condition is equal to the combinatorial number of tuples composed of* $(2n+1)$ *integers* $(\gamma_1, \ldots, \gamma_n, p_0, p_1, \ldots, p_n)$ *such that*

$$\sum_{l=0}^{n} p_l = m, \quad k = \gamma_n \geq \gamma_{n-1} \geq \cdots \geq \gamma_1 \geq \frac{1}{2}\left(\sum_{l=0}^{n} |p_l| - |m|\right).$$

Regarding its eigenvalues, there is a slight difference between $\lambda_m^{(k)}$ and $\widetilde{\lambda}_m^{(k)}$ by $n^2/\ $, but this is constant under the change of k and m. Therefore, the gap of eigenvalues caused by the change of k, its quantum number, is equal to each other. It is more important that multiplicity is exactly equal to each other because if any difference is, that numbers could be out of sense.

That is why we would like to verify that each solution method having performed by Kuwabara and Yoshioka is equivalent to each other on computing a spectrum which leads to the concept of quantization. For the purpose of this, we actually check that the combinatorial number of tuples composed of $(2n+1)$ integers $(\gamma_1, \ldots, \gamma_n, p_0, p_1, \ldots, p_n)$ coincides completely with the expression for multiplicity of $\lambda_m^{(k)}$ in the Proposition 1.

In Section 2, we just do mathematics of counting to obtain the combinatorial number of tuples. Then in Section 3, we prove the combinatorial number equals the expression for the multiplicity of $\lambda_m^{(k)}$.

2. Problems of Counting Lattice Points

In this section, we count up the combinatorial number of tuples composed of $(2n+1)$ integers $(\gamma_1, \ldots, \gamma_n, p_0, p_1, \ldots, p_n)$ shown in Theorem 2.

First, it should be noted that we apply the following definition throughout this paper.

Definition 3.

$$\binom{a}{b} := 0 \quad \text{if } a < b \text{ or } b < 0$$

Then we obtain the following formulas to count the number of lattice points composed of $(n+1)$ integers $(x_1, \dots, x_{n+1})$.

Lemma 4. *Let n be a natural number. The number of integer lattices $(x_1, \dots, x_{n+1})$ satisfying the following equations: for $m, i \quad \mathbb{Z}$,*

$$\sum_{j=1}^{n+1} x_j = m, \quad \frac{1}{2}\left(|x_1| + \cdots + |x_{n+1}| - |m|\right) = i,$$

is given by the following expressions:

$$\begin{cases} \binom{|m|+n}{|m|}, & i = 0, \\ \displaystyle\sum_{t=1}^{\min\{i,n\}} \binom{n+1}{t}\binom{i-1}{i-t}\binom{|m|+i+n-t}{|m|+i}, & i > 0, \\ 0, & i < 0. \end{cases}$$

Next, we consider the number of integer lattices $(\gamma_1, \dots, \gamma_n, p_0, p_1, \dots, p_n)$ satisfying the following equations for $m \quad \mathbb{Z}$ and $k = 0, 1, 2, \dots,$

$$\sum_{l=0}^{n} p_l = m, \quad \frac{1}{2}\left(|p_0| + \cdots + |p_n| - |m|\right) \leq \gamma_1 \leq \cdots \leq \gamma_n = k. \tag{2}$$

When we fix the integer $\frac{1}{2}\left(|p_0| + \cdots + |p_n| - |m|\right)$ at $i \geq 0$, we have the number of integer lattices $(\gamma_1, \dots, \gamma_n)$ satisfying the inequality $i \leq \gamma_1 \leq \cdots \leq \gamma_{n-1} \leq \gamma_n = k$ as follows, because it is equivalent to distribute $(n-1)$ things among $(k-i+1)$ people under the condition that there may be some people who can receive nothing:

$${}_{k-i+1}H_{n-1} = {}_{k-i+n-1}C_{n-1} = \binom{k-i+n-1}{n-1} = \binom{k-i+n-1}{k-i}.$$

Then we replace the integer p_l with x_j $(j = l+1)$. By the previous Lemma 4, the number of integer lattices $(p_0, \dots, p_n)$ satisfying the following equations: for $m \quad \mathbb{Z}$ and $i = 0, 1, 2, \dots,$

$$\sum_{l=0}^{n} p_l = x_1 + \cdots + x_{n+1} = m,$$

$$\frac{1}{2}\left(|p_0| + \cdots + |p_n| - |m|\right) = \frac{1}{2}\left(|x_1| + \cdots + |x_{n+1}| - |m|\right) = i$$

is given by

$$\begin{cases} \dbinom{|m|+n}{|m|}, & i=0, \\ \displaystyle\sum_{t=1}^{\min\{i,n\}} \binom{n+1}{t}\binom{i-1}{i-t}\binom{|m|+i+n-t}{|m|+i}, & i>0. \end{cases}$$

Thus we obtain the number of integer lattices $(\gamma_1,\ldots,\gamma_n,p_0,p_1,\ldots,p_n)$ satisfying (2) as follows:

$$\begin{cases} \dbinom{k+n-1}{k}\dbinom{|m|+n}{|m|}, & i=0, \\ \dbinom{k-i+n-1}{k-i} \displaystyle\sum_{t=1}^{\min\{i,n\}} \binom{n+1}{t}\binom{i-1}{i-t}\binom{|m|+i+n-t}{|m|+i}, & 1\le i\le k. \end{cases} \tag{3}$$

When k is equal to 0, i is obviously equal to 0. Its converse proposition is, however, untrue. Namely, $k=0 \Rightarrow i=0$ is true. But when $k\neq 0$, i is not necessarily a natural number, that is, $i=0,1,\ldots,k$. As a result of using (3), we can write the number of integer lattices $(\gamma_1,\ldots,\gamma_n,p_0,p_1,\ldots,p_n)$ satisfying (2) by the following proposition.

Proposition 5. *Let n be a natural number. Then the number of integer lattices $(\gamma_1,\ldots,\gamma_n,p_0,p_1,\ldots,p_n)$ satisfying the following equations: for $m \quad \mathbb{Z}$ and $k=0,1,2,\ldots,$*

$$\sum_{l=0}^{n} p_l = m, \quad \frac{1}{2}\left(|p_0|+\cdots+|p_n|-|m|\right) \le \gamma_1 \le \cdots \le \gamma_n = k$$

is given by the following expressions:

$$\begin{cases} \dbinom{|m|+n}{|m|}, & k=0, \\ \dbinom{k+n-1}{k}\dbinom{|m|+n}{|m|} \\ \quad +\displaystyle\sum_{i=1}^{k}\binom{k-i+n-1}{k-i}\sum_{t=1}^{\min\{i,n\}} \binom{n+1}{t}\binom{i-1}{i-t}\binom{|m|+i+n-t}{|m|+i}, & k\in\mathbb{N}. \end{cases} \tag{4}$$

3. A Proof of the Equality between Two Numbers

In this section, we show that the number of lattice points obtained in Proposition 4 is equivalent to the expression (1) which is for the multiplicity given by Kuwabara.

In the case of $k=0$, (1) gives the same number as (4) as follows:

$$\binom{|m|+n}{|m|}\binom{n}{0} - \binom{|m|+n-1}{|m|-1}\binom{n-1}{-1} = \binom{|m|+n}{|m|}$$

because $\binom{n-1}{-1} := 0$ due to Definition 3. Then the case where $k \quad \mathbb{N}$ remains, and we obtain the following proposition.

Proposition 6. *For $n, k \quad \mathbb{N}$ and $m \quad \mathbb{Z}$,*

$$\begin{aligned}
&\binom{k+|m|+n}{k+|m|}\binom{k+n}{k} - \binom{k+|m|+n-1}{k+|m|-1}\binom{k+n-1}{k-1} \\
&= \binom{k+n-1}{k}\binom{|m|+n}{|m|} \\
&\quad + \sum_{i=1}^{k} \binom{k-i+n-1}{k-i} \sum_{t=1}^{\min\{i,n\}} \binom{n+1}{t}\binom{i-1}{i-t}\binom{|m|+i+n-t}{|m|+i}.
\end{aligned}$$

Proof. We intend to find the function of k ($k \quad \mathbb{N}$) denoted by $F(k)$ which satisfies the following simultaneous equations:

$$\begin{cases}
lF(k+1) - F(k) = \binom{k+n-1}{k}\binom{|m|+n}{|m|} + \sum_{i=1}^{k} \binom{k-i+n-1}{k-i} C_{i,n}, \\
F(1) = \binom{|m|+n}{|m|},
\end{cases}$$

where $C_{i,n}$ is the constant with regard to k (depending on i and n) defined by

$$C_{i,n} := \sum_{t=1}^{\min\{i,n\}} \binom{n+1}{t}\binom{i-1}{i-t}\binom{|m|+i+n-t}{|m|+i}. \tag{5}$$

Then we find

$$F(k+1) = \binom{k+n}{n}\binom{|m|+n}{|m|} + \sum_{i=1}^{k} \binom{k+n-i}{n} C_{i,n}\,. \tag{6}$$

On the other hand, we certainly imply that $F(k)$ is equal to

$$\binom{k+|m|+n-1}{k+|m|-1}\binom{k+n-1}{k-1},$$

then using the following formula (Vandermonde convolution, see [2]):

$$\binom{N_1+N_2}{n} = \sum_{\nu=0}^{\min\{n,N_1\}} \binom{N_1}{\nu}\binom{N_2}{n-\nu},$$

we consider the following product of binomial coefficients:

$$\begin{aligned} F(k+1) &= \binom{(k+1)+|m|+n-1}{(k+1)+|m|-1}\binom{(k+1)+n-1}{(k+1)-1} \\ &= \binom{k+n}{n}\binom{k+|m|+n}{n} \\ &= \binom{k+n}{n}\sum_{\nu=0}^{\min\{n,k\}}\binom{k}{\nu}\binom{|m|+n}{n-\nu} \\ &= \binom{|m|+n}{|m|}\binom{k+n}{n} + \binom{k+n}{n}\sum_{\nu=1}^{\min\{n,k\}}\binom{k}{\nu}\binom{|m|+n}{n-\nu}. \end{aligned} \tag{7}$$

Comparing (7) with (6), the following lemma arises, that is, this proof should be complete with this lemma.

Lemma 7 (Main Result).

$$\sum_{i=1}^{k}\binom{k+n-i}{n}C_{i,n} = \binom{k+n}{n}\sum_{\nu=1}^{\min\{n,k\}}\binom{k}{\nu}\binom{|m|+n}{n-\nu}, \tag{8}$$

where $C_{i,n}$ is defined by (5)*.*

We are sorry to only give an idea of the proof for this lemma because of a space restriction. Making use of Vandermonde convolution and Jensen's Formula (see [3, Vol. 5.PDF 19p (4.1)]):

$$\sum_{z=0}^{N}\binom{x+yz}{z}\binom{r-yz}{N-z} = \sum_{z=0}^{N}\binom{x+r-z}{N-z}y^z,$$

we can show that both sides of (8) are equal to

$$\sum_{\nu=1}^{k}\binom{|m|+n}{n-\nu}\binom{n+\nu}{n}\binom{k+n}{k-\nu}.$$

Thus, Lemma 7 and Proposition 6 are proved simultaneously. □

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Applications of the Fundamental Theorems of Projective and Affine Geometry in Physics

Patrick Moylan

Dedicated to Daniel Sternheimer on the occasion of his 80th birthday in 2018

Abstract. The closely related fundamental theorems of projective and affine geometry are keys to understanding some important but seemingly unrelated topics in mathematical physics. Specifically, they can be used in proofs of Wigner's theorem on ray correspondences in quantum mechanics and also to establish the scale extended Poincaré group as the basic causality preserving symmetry group of special relativity. We describe these two theorems and show their roles in establishing the just mentioned results.

Mathematics Subject Classification (2000). Primary 51N10, 51N15; Secondary 83A05, 81Pxx.

Keywords. Wigner's theorem, Alexandrov–Zeeman Theorem.

1. Introduction

The fundamental theorem of affine geometry, a proof of which can be found in [1], states that any isomorphism of an affine space over a skew field K which takes lines to lines and preserves parallelism is, up to a translation, necessarily a linear transformation. The fundamental theorem of projective geometry asserts that every collineation of a projective space $\mathcal{P}(V)$ is induced by a semilinear map of the associated vector space V. A proof of this theorem for finite dimensions is given in [2]. These theorems have important applications to physics. The latter has been used to obtain proofs of Wigner's theorem in quantum mechanics [3–5] and the former can be used to show how causality implies that the scale-extended Poincaré group is the basic symmetry group of special relativity [6–9]. In this article we describe these important theorems and show their central importance in establishing the just stated applications to mathematical physics.

2. The Fundamental Theorems of Affine Geometry and Projective Geometry

These two theorems hold in a very general setting. The number field over which the affine or projective space is defined can be a skew field $\mathbb{F}$ such as the quaternions. The dimension of the affine space or projective space can even be infinite. An affine space is a set A on which a vector space V acts freely and transitively. The dimension of an affine space A is defined to be the same as the dimension of the vector space which acts on A. We denote the action of V on A by addition: $a \to a + v = v + a \quad A$ if $a \quad A$ and $v \quad V$. Given two points a, b in A, the unique vector $\overrightarrow{ab}$ specified by $a + \overrightarrow{ab} = b$ is called the displacement vector from a to b. If we choose a reference point 0 (called the origin) in A, then A can be identified with V by the correspondence $v \to 0 + v$. Thus a vector space V is itself an affine space under translation of vectors.

By an affine map from an affine space A into another affine space A' we mean a map $\varphi : A \to A'$ for which the correspondence

$$\overrightarrow{ab} \mapsto \overrightarrow{\varphi(a)\varphi(b)}$$

gives a well-defined linear map f of V into V'. If φ is bijective, it is called an isomorphism of affine spaces and two affine spaces are said to be isomorphic. When relevant affine spaces are furnished with reference points (origins), an affine map φ takes the form

$$V \ni v \mapsto 0 + v \mapsto \varphi(0 + v) - 0' = f(v) + \varphi(0) - 0' \quad V'$$

which can be viewed as the composition of the linear map f which takes v to $f(v)$ followed by a translation by $\varphi(0) - 0' \quad V'$. Two affine spaces are isomorphic if and only if they have the same dimension. Consequently any finite-dimensional affine space A is isomorphic to $\mathbb{K}^n$ where $n = \dim A$.

A mapping $f : V \to V'$ is additive if $f(v + w) = f(v) + f(w)$. An additive mapping $f : V \to V'$ is said to be **semilinear** if there is an automorphism $\varphi : \mathbb{K} \to \mathbb{K}$ of $\mathbb{K}$, which fixes the identity element in $\mathbb{K}$ (i.e. $\varphi(1) = 1$), so that $f(\ell v) = \varphi(\ell) f(v)$ for $\ell \quad \mathbb{K}$ and $v \quad V$. In the mathematical physics literature, φ is frequently called a twist.

Theorem 1 (The Fundamental Theorem of Affine Geometry). *Let γ be a $1 \to 1$ transformation of an affine space A over a skew field $\mathbb{K}$ onto itself which maps straight lines onto straight lines and preserves parallelism. (Preserving parallelism means that if two lines are parallel then their image under γ again consists of two parallel lines with all points on one of the image lines necessarily being images of just one of the parallel lines in the domain, since γ takes lines to lines.) Let γ_0 be the corresponding $1 \to 1$ transformation of V. Then γ_0 is of the form*

$$\gamma_0 : \Sigma \lambda_i v_i \to \Sigma \varphi(\lambda_i) \Lambda(v_i)$$

where $\Lambda : V \to V$ is additive (i.e. a linear mapping) and φ is an automorphism of $\mathbb{K}$. (v_i is a base of V.) In other words, γ_0 is a semilinear bijective mapping of the associated vector space V.

Now to the fundamental theorem of projective geometry: the projective space $\mathbb{P}(V)$ associated to a $\mathbb{K}$-vector space V over a field (or more generally skew-field, i.e. non-commutative field such as the quaternions) is the set of one-dimensional subspaces of V. Let 0 denote the zero vector (origin) in V. There is a surjective mapping $V^\times = V \setminus 0 \ni v \to [v] = \mathbb{K}v \quad \mathbb{P}(V)$ and $\mathbb{P}(V)$ is identified as the quotient space $V^\times/\mathbb{K}^\times$. Now let X be another vector space. If $f : X \to V$ is a linear mapping which is injective, then it induces an injective map $[f]$ on $\mathbb{P}(X)$ by $\mathbb{P}(X) \ni \mathbb{K}x \to \mathbb{K}f(x) \quad \mathbb{P}(V)$. In particular, if X is a subspace of V, $\mathbb{P}(X) \subset \mathbb{P}(V)$. A subset of the form $\mathbb{P}(X)$ with $X \subset V$ is called a projective subspace of $\mathbb{P}(V)$ with the dimension of $\mathbb{P}(X)$ defined to be $\dim X - 1$. Projective subspaces of dimension one and two are referred to as projective lines and projective planes, respectively.

$\mathbb{K}^\times$ is a group under multiplication and V is a $\mathbb{K}^\times$ module, a left or right $\mathbb{K}^\times$ module, if $\mathbb{K} = \mathbb{Q}$. Working in $\mathbb{C}$ we have a representation of $\mathbb{C}^\times$ on V, and we can consider orbits of points in V under the action of the group $\mathbb{K}^\times = \mathbb{C}^x$. Let us define a **ray** to be an orbit of the form $v \to e^{i\alpha}v$ on $V^\times$. In other words, a ray is the set

$$[v] = \{e^{i\alpha}v \,|\, \alpha \quad \mathbb{C}\,,\ 0 \neq v \quad V\}.$$

Clearly, ray space is just complex projective space, $\mathbb{P}(V)$.

When V is a Hilbert space over $\mathbb{C}$, we can define a **unit ray**. It is a ray with norm one, which means $\|v\| = 1$ and also $\alpha \quad \mathbb{R}$ so that $e^{i\alpha}$ is a phase factor, i.e. $|e^{i\alpha}| = 1$. The set of all rays is called **ray space**. (Technically what we called a ray in the previous paragraph should probably be called a *generalized* **ray**, since in physics a **ray** is usually defined as an orbit of a unit vector whose representatives all have norm one.)

Given two points $[a]$ and $[b]$ of a projective space $\mathbb{P}(V)$, we denote by $[a] \vee [b]$ the subspace generated by $[a]$ and $[b]$. If $[a] \neq [b]$ this is the line containing $[a]$ and $[b]$. Specifically,

$$[a] \vee [b] := \{a + b \quad V | a \quad [a], b \quad [b]\}.$$

Three points $[a]$, $[b]$ and $[c] \quad \mathbb{P}(V)$ are called collinear if $[c] \quad [a] \vee [b]$.

A bijective map $\mathbb{T} : \mathbb{P}(V) \to \mathbb{P}(W)$ is called a collineation if $\mathbb{T}$ preserves collinearity, i.e. $\forall$ $[a]$, $[b] \quad \mathbb{P}(V)$, we have

$$\mathbb{T}([a] \vee [b]) = \mathbb{T}([a] \vee \mathbb{T}([b]).$$

Theorem 2 (The Fundamental Theorem of Projective Geometry). *Let $\mathbb{P}(V)$ and $\mathbb{P}(W)$ be projective spaces of dimension $n \geq 2$ and let $\mathbb{T} : \mathbb{P}(V) \to \mathbb{P}(W)$ be a collineation. Then $\mathbb{T}$ is of the form $[T]$ where $T : V \to W$ is a (unique up to scalar multiplication) semilinear, bijective mapping from V to W compatible with $\mathbb{T}$. (A map $T : V \to W$ is* ***compatible*** *with $\mathbb{T}$ provided $[Tv] = \mathbb{T}([v])$ for all $v \quad V$).*

3. Applications

Wigner's theorem is usually formulated as follows:

Theorem 3 (Wigner). *Let $\mathfrak{H}$ be a Hilbert space over $\mathbb{C}$ with inner product $(\cdot,\cdot)$. Let $S : \mathfrak{H} \to \mathfrak{H}$ be a bijective map such that*

$$|(Sx, Sy)| = |(x, y)| \tag{1}$$

for all x, y $\quad\mathfrak{H}$. Then S is phase-equivalent to either to a unitary or antiunitary operator.

(By phase equivalent we mean the following: two bijective maps $S : V \to V$ and $T : V \to V$ of a complex vector space V are phase-equivalent if there exists a scalar valued function $\rho(x)$ on V such that 1) $|\rho(x)| = 1$ for all x $\quad V$ and 2) $Tx = \rho(x)Sx$ for all x $\quad V$. By antiunitary operator we mean an antilinear map T of $\mathfrak{H}$ onto $\mathfrak{H}$ such that $T\lambda v = \bar{\lambda}Tv$.)

Clearly the S of Wigner's theorem specifies a unique bijection $\mathbb{S}$ on the projective space $\mathbb{P}(\mathfrak{H})$ defined by $\mathbb{S}([v]) = [Sv]$ for v $\quad\mathfrak{H}$. Since, from Eq. (1), S takes orthonormal bases to orthonormal bases, it is not difficult to see that $\mathbb{S}$ is a collineation. Now apply the fundamental theorem of projective geometry to obtain a semilinear operator $\widetilde{S}$ on $\mathfrak{H}$ which is phase-equivalent to S. Finally use Eq. (1) to show that $\widetilde{S}$ is a unitary or antiunitary operator. We refer the reader to especially Keller [3] for the details (cf. also [4, 5]).

Now to the applications to special relativity. We assume our space-time is an affine space A. An **inertial transformation** is a $1 \to 1$ transformation γ of A onto itself which takes one inertial frame into another. We require that γ and γ^{-1} map straight lines onto straight lines and preserves parallelism. The reason for the requirement of parallelism can be seen as follows: according to the principle of relativity, we cannot distinguish two different inertial frames from one another in any absolute sense, i.e. no one inertial frame can have any preferred property distinguishing it from any other. Thus, if two straight lines, describing the world lines of two moving or stationary particles, are judged parallel according to any one inertial observer, then they must also be judged parallel to all other inertial observers.

From the previous paragraph, it is clear that an inertial transformation γ satisfies the hypotheses of the fundamental theorem of affine geometry. Thus γ is, up to a translation, a semilinear bijective mapping of space-time onto itself. Furthermore, since the only field automorphism of $\mathbb{R}$ is the identity, we have, in fact, shown that any inertial transformation must necessarily be, up to a translation, a linear transformation of the underlying vector space V of space-time. Additionally, since all finite dimensional vector spaces are isomorphic, we have proven for n dimensional space-time that:

Theorem 4. *Any inertial transformation of n dimensional space-time must necessarily be, up to a translation, a linear transformation of $\mathbb{R}^n$ onto itself.*

Our Theorem 4 simplifies the proofs in such papers as [7, 9] and [10] establishing the scale-extended Poincaré group as the maximal transformation group of space-time connecting different inertial observers to one another and preserving causality. We now illustrate an approach leading to this result in the case in which space-time is assumed to be two dimensional and making use of Theorem 4. Proper linear transformations of $\mathbb{R}^2$ are elements of $GL^+(2,\mathbb{R})$. Also $\mathfrak{gl}(2,\mathbb{R})$ is a trivial central extension of $\mathfrak{sl}(2,\mathbb{R})$ with the (commutative) center of the extension being just infinitesimal scale transformations. Thus, according to Theorem 4, we have, up to scale transformations and translations in space-time, both of which preserve causality (see below), that the admissible transformations from one inertial frame to another are precisely the elements of $SL(2,\mathbb{R})$. We have the following:

Proposition 1. *A continuous one-parameter subgroup of* $SL(2,\mathbb{R})$ *is conjugate under* $GL^+(2,\mathbb{R})$ *to one of the following subgroups* [11]*:* $\left(\Omega = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right)$

$$A = \left\{ a(s) = \begin{pmatrix} \operatorname{ch} s & \operatorname{sh} s \\ \operatorname{sh} s & \operatorname{ch} s \end{pmatrix} \middle| t \quad \mathbb{R} \right\} = \left\{ \Omega \exp \begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix} \Omega^{-1} \middle| s \quad \mathbb{R} \right\},$$

$$N = \left\{ \begin{pmatrix} 1+\frac{v}{2} & \frac{v}{2} \\ -\frac{v}{2} & 1-\frac{v}{2} \end{pmatrix} \middle| t \quad \mathbb{R} \right\} = \left\{ \Omega \exp \begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix} \Omega^{-1} \middle| v \quad \mathbb{R} \right\},$$

$$K = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \middle| \theta \quad \mathbb{R} \right\} = \left\{ \Omega \exp \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \Omega^{-1} \middle| \theta \quad [0, 2\pi) \right\}.$$

Furthermore, any element g *of* $SL(2,\mathbb{R})$ *can be uniquely written as* $g = kan$ *with* $k \quad K$*,* $a \quad A$ *and* $n \quad N$ *(Iwasawa decomposition)* [12]*.*

Space-time should be endowed with a **causal structure**. In $\mathbb{R}^n$ ($n-1$ spatial dimensions) causality is defined as follows: an event (t,x) can influence an event (t',x') (with $t'>t$) $\iff$ the two points are connected by a straight world line, the speed of which can never be infinite, i.e. $\left\| \frac{x'-x}{t'-t} \right\| < \infty$. If we insist that there is some finite limiting speed c, then we have:

$$(t,x) \prec (t',x') \iff c\cdot(t'-t) \geq \|x'-x\|. \tag{2}$$

This causality relation ($\prec$) endows $\mathbb{R}^n$ with a family of convex cones and defines a causal structure. Clearly, $\prec$ is preserved under scale transformations, translations in both space and time and rotations and Lorentz transformations. The Alexandrov–Zeeman theorem states that the inverse is also true: the set of all possible bijections of $\mathbb{R}^n$ which preserves the causality relation is precisely the scale-extended Poincaré group. The Alexandrov–Zeeman theorem is quite general and independent of our assumptions regarding inertial transformations.

Now to finish our proof of the Alexandrov–Zeeman result for two dimensional space-time. We stress that ours is a simpler proof than Alexandrov's and Zeeman's, but it involves more assumptions: we demand, in addition to preservation of the causality relation, that our transformations also satisfy the hypothesis of Theorem 1. These additional assumptions immediately lead us, in the two dimensional case, to $GL^+(2,\mathbb{R})$ and hence to the subgroups of Proposition 1. It is easy to see that of the subgroups listed in Proposition 1 only the elements of A are causality preserving transformations according to the definition of causality specified in Eq. (2) (with $c = 1$).

The parabolic subgroup $\Omega^{-1}N\Omega$ is the limit of the subgroup A under conjugation. It is obtained from A by conjugation with $C = \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}$ $(c \quad \mathbb{R}^+)$ followed by taking appropriate limits. Specifically, we take $a(s) \to C^{-1}a(s)C$ followed by taking the limits $c \to \infty$, $s \to 0$ in such a way that $c\sinh(s) \to v$. With Eq. (2) having the limiting value $c = \infty$, elements of this conjugacy limit, i.e. of $\Omega^{-1}N\Omega$, preserves Eq. (2). It leads to Galilean relativity. A and its conjugates under C leads to special relativity. Thus we obtain special relativity or Galilean relativity solely from the requirement of causality and reasonable assumptions about invariance properties of world lines under transformations between inertial frames together with considerations about limiting cases.

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Modeling the dynamics of a charged drop of a viscous liquid

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Abstract. We consider a mathematical model of motion, conducting drops of a viscous fluid, immersed in a dielectric viscous fluid of infinite length and changing under the influence of capillarity and electrostatic repulsion. Using the equations of hydrodynamics and electromagnetism and a number of physically realistic assumptions, the problem is reduced to a system of partial differential equations. The solution of such a system of equations is particularly difficult. Computer simulation in the COMSOL Multiphysics environment allowed us to obtain changes of the shape the charged of drop with time.

Mathematics Subject Classification (2000). Primary PACS: 47.55.Ca; Secondary 76T10.

Keywords. conductive liquid drop, dielectric viscous, capillarity, electrostatic force, computer simulation.

1. Introduction

The modern interest in electro hydrodynamics and, in particular, in the evolution of charged droplets of liquids, the formation of the stability of these droplets and the formation of finite-time features on the free surface are associated with many applications of science and technology. The tasks of electro hydrodynamics of micro and nano-scales attract great attention of researchers in connection with a wide field of application, mainly in nano- and biotechnologies. In particular, fuel spraying, coating, inkjet printing, drip cooling, chemical treatment of plants and many other industries [1]. Simulation of Stokes flows in various areas, as well as laboratory studies on the dynamics of emulsion droplets in microchannels in a wide range of values of various parameters affecting the physical properties of the entire system as a whole, have a very limited number of solutions to such problems, costly and difficult. Computer simulation allows one to plan, partially replace and significantly complement the experiments [2]. In this paper, we study the evolution of electrically charged liquid droplets moving under the action of surface tension and electrostatic forces. We are especially interested in the stability of these drops and the formation of features on the free surface. These features have the form of conical tips on the surface of a drop, where the curvature of the surface and the velocity field of the liquid diverge at a certain time.

2. The mathematical model of the motion of a charged drop of liquid

Consider a drop of liquid D with a viscosity μ_{in} suspended in an infinite liquid with viscosity μ_{out} (Fig. 1). Suppose the drop size is on the order of a few micrometers. Then we can assume that the Reynolds number

$$Re = \frac{\rho u_0 R}{\mu} = 1,$$

where u_0 is the characteristic velocity of the fluid, R is the characteristic size of the drop, ρ is the density of the liquid, and μ is the viscosity of the liquid.

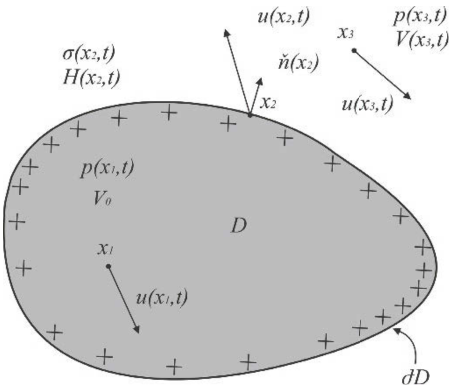

FIGURE 1. Charged drop of liquid

On the other hand, the size of the drop is large enough to ensure the fulfillment of the continuity hypothesis, in accordance with which the aggregate of motion of fluid molecules can be considered as a continuum. Note also that the effect of evaporation on the evolution of a drop is not taken into account. The time scale of the movement of the free border is of the order of milliseconds, which is clearly not enough for significant mass loss due to evaporation. The liquid inside the drop is the ideal electrical conductor, while the liquid outside the drop is the ideal electrical insulator. In addition, the drop has an electric charge Q. The motion of both internal and external fluid we write in accordance with the Stokes approximation of the Navier–Stokes equations [3]:

- The equation of conservation of motion of the liquid in the droplet

$$\mu_{\text{in}}\nabla^2\vec{u}(\vec{x},t) = \nabla p(\vec{x},t), \quad \vec{x} \quad \mathbf{D}; \tag{1}$$

- The equation of conservation of motion of fluid outside a drop

$$\mu_{\text{out}}\nabla^2\vec{u}(\vec{x},t) = \nabla p(\vec{x},t), \quad \vec{x} \;/\; \mathbf{D}; \tag{2}$$

- The continuity equation

$$\nabla\vec{u}(\vec{x},t) = 0, \tag{3}$$

where u – is the velocity field, p is the pressure in the fluid.

From equations (1)–(3) it follows that the pressure p is a harmonic function outside the drop boundary:

$$\nabla^2 p(\vec{x},t) = 0, \quad \vec{x} \;/\; \partial\mathbf{D}. \tag{4}$$

The movement of the most free drop boundary depends on time and is determined by the kinematic condition:

$$\frac{d\vec{x}}{dt} = \vec{u}(\vec{x},t), \quad \vec{x} \quad \partial\mathbf{D}. \tag{5}$$

Since the fluid inside the droplet is conductive, the charges in it repel each other and move toward the free boundary, and since they cannot penetrate the non-conductive external fluid, they are ultimately distributed along the free D boundary with a surface charge density σ and create electric field with electric potential V. The electric potential inside the drop is constant, i.e.,

$$V(\vec{x},t) = V_0, \quad \vec{x} \quad \mathbf{D}, \tag{6}$$

and satisfies the Laplace equation outside the drop D.

At the free boundary, the electric potential V is

$$\nabla^2 V(\vec{x},t) = 0, \quad \vec{x} \;/\; \mathbf{D}, \tag{7}$$

which is continuous, but the normal derivative $\frac{\partial V}{\partial n}$ suffers a discontinuity. The jump of the normal derivative and the surface charge density σ on the free boundary fulfills the relation:

$$\left[\frac{\partial V}{\partial n}\right]_{\partial D} = -\frac{\sigma}{\varepsilon_0}, \tag{8}$$

where ε_0 is the dielectric constant of the environment. The density of surface charge σ also satisfies the condition

$$\int_{\partial D} \sigma ds = Q \tag{9}$$

and by virtue of the law of conservation of electric charge Q is constant in time. For the velocity field, we write the following boundary condition on the free boundary:

$$[S\vec{n}]_{\partial D} = \left(2\gamma H - \frac{\varepsilon_0}{2}\left(\frac{\partial V}{\partial n}\right)^2\right)\vec{n} \text{ or } [S\vec{n}]_{\partial D} = \left(2\gamma H - \frac{\sigma^2}{2\varepsilon_0}\right)\vec{n}, \qquad (10)$$

where γ surface tension coefficient, H average curvature, n outer normal vector to the free boundary, S stress tensor, which is defined as follows:

$$S_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i = 1, 2, 3. \qquad (11)$$

Note that this boundary condition relates the fluid velocity field to the electrostatic field (through the density of surface charge σ). An interesting dynamics arises due to the interaction of two opposite forces acting at the interface between two liquids: capillary forces and electrostatic forces in the expression (10). While the capillary forces tend to make the drop as smooth and spherical as possible, the electric forces tend to increase any disturbance in the shape of the drop, as the charges accumulate in parts of the free border with high curvature and strongly deform the surface at these points (Fig. 2).

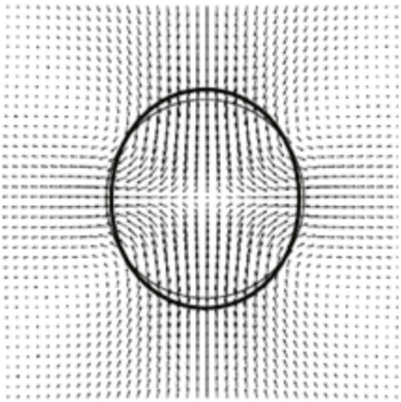

FIGURE 2. The change of the drop's border

3. Simulation of a charged drop of liquid in COMSOL Multiphysics

Computer simulation in the COMSOL Multiphysics environment allows you to explore physical phenomena that can be characterized as electromagnetism, structural mechanics, acoustics, hydrodynamics, thermal and chemical reactions, as well as partial differential equations. The COMSOL Multiphysics software module was selected to study and analyze the dynamics of a charged drop. The package interface is selected taking into account the considered mathematical model of the movement of the border of a charged liquid drop and defined above by the following equations: (1)–(3) and (5)–(7). But at the ∂D boundary (Fig. 3), the normal derivative of the electric potential $\frac{\partial V}{\partial n}$ is a discontinuous function. The jump of the normal derivative of the electric potential and the density of surface charge σ on the free boundary are related and are represented by the expressions (8) and (10).

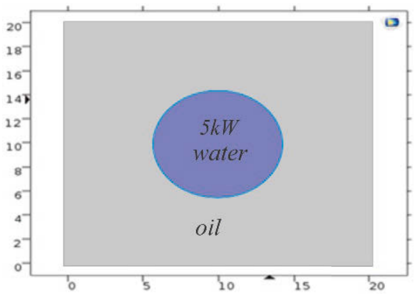

FIGURE 3. Geometric parameters of the form of a charged drop and the external environment of oil.

Using the COMSOL Multiphysics package, we will model the dynamics of changes in the shape of a charged drop in oil, in a medium with a more viscous fluid. As an example, consider the change in the shape of a charged droplet with a charge of 5 kW, placed in oil. The initial speed of the movement of the oil is zero. At the initial moment, a laminar two-phase flow is specified, the interfaces of the phase field are set by the equations of fluid motion in accordance with the Navier–Stokes equations:

$$\frac{p\partial u}{\partial t} + p(u\nabla)u = \nabla[-pl + \mu(\nabla u + (\nabla u)^T)] + F_{st} + pg + F, \quad \nabla u = 0.$$

Here u is the velocity, ρ density, μ viscosity, p pressure, I unit vector, g acceleration, F_{st} surface stress, and F additional force. The electrostatic interface is set by the equation for V (electrostatic potential): $-\nabla(\epsilon_0\epsilon\ \nabla V) = 0$, where ϵ_0 is vacuum permeability, ϵ relative conductivity.

The program allows you to automatically set the equations described earlier. For a two-phase flow, you must specify the power. Electric power is determined by $F = \nabla T$ where the Maxwell tensor: $T = ED^T - \frac{1}{2}(ED)I$. Here, E is the electric field, D the electric shift: $E = \nabla V, D = \varepsilon_0\varepsilon\ E$. In this example, the stress tensor is two-dimensional $2D$, therefore:

$$T = \begin{bmatrix} T_{xx} & T_{xy} \\ T_{yx} & T_{yy} \end{bmatrix} = \begin{bmatrix} \varepsilon_0\varepsilon\ E_x^2 - \frac{1}{2}\varepsilon_0\varepsilon\ (E_x^2 + E_y^2) & \varepsilon_0\varepsilon\ E_xE_y \\ \varepsilon_0\varepsilon\ E_yE_x & \varepsilon_0\varepsilon\ E_y^2 - \frac{1}{2}\varepsilon_0\varepsilon\ (E_x^2 + E_y^2) \end{bmatrix}.$$

The components of the electric field are calculated using an electrostatic interface. To determine the viscosity of a fluid, when a charged drop of water spreads in a more viscous medium in oil, the following formula can be used: $\mu = \frac{gD^2(p_B p_M)}{18u}$. As we know, $p_B = 0.98$ g/cm , $p_M = 0.89$ g/cm . The results of computer simulation in Fig. 4 showed that, using the COMSOL Multiphysics package, in all tests there is a gradual change in the shape of a charged drop in oil. The program code adequately models the position of the phase interface.

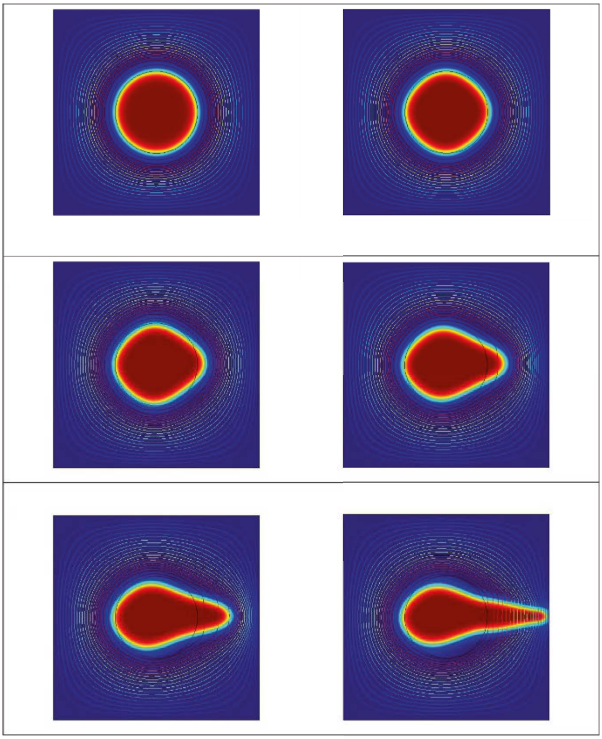

FIGURE 4. Dynamics of change of the free boundary of the drop at $t = 0.01$ s, $t = 0.2$ s, $t = 0.35$ s, $t = 0.\ 3$ s, $t = 0.\ 7$ s.

4. Conclusion

The linear stability of a family of solutions arising due to the perturbation of a sphere of radius R (which is an equilibrium solution) was analyzed [3–5]. According to the results of the work, one can see how the stability of a drop depends on the values of charge, volume, surface tension and viscosity, as well as on the shape of the drop. As a result, a computer model of a charged drop of liquid based on the COMSOL Multiphysics software module was created on the basis of the presented mathematical model. Computer and mathematical models adequately reflect the dynamics of changes in the boundary of a charged drop of liquid and show the position of the phase boundary.

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The orthogonal systems of functions on lattices of SU$(n+1)$, $n<\infty$

Mariia Myronova and Marzena Szajewska

Abstract. The definitions of orbit functions, their orthogonality relations, congruence classes and decomposition matrices are recalled. The orthogonality of the symmetric C- and antisymmetric S-orbit functions, which are given on the fundamental region F_M of the weight lattice, for simple Lie group $SU(n+1)$ of any rank n is defined. The splitting of the weight lattice of A_n into congruence classes is shown.

Mathematics Subject Classification (2000). 20F40,22E70,42B05.

Keywords. C-functions, S-functions, orthogonality, congruence classes, decomposition matrices.

1. Introduction

A simple Lie group of type $SU(n+1)$ is generally used to describe continuous symmetry operations such as rotations about the axes. In general, generators of rotation about different axes do not commute, but they can form a Lie algebra which closes under the commutation.

There are two types of orthogonal functions (symmetric and antisymmetric orbit functions, also refereed as C- and S-functions) that are investigated in this paper [3,4]. Due to their remarkable properties, these functions can be treated as special functions. We define C- and S-functions for a simple Lie algebra of type A_n. Both systems of functions are orthogonal on the fundamental region F_M of the weight lattice of $SU(n+1)$, or, equivalently, on the weight lattice of the Lie algebra whose root system is A_n. Here the region F_M is a simplex in $\mathbb{R}^n$. However, the weight lattice of A_n can be split into $(n+1)$ congruence classes of points, hence we can obtain the orthogonality of the special functions.

In this paper we also present a structure of decomposition matrices for the Fourier transforms of some data sampled on the lattices. These matrices are the square matrices of the size determined by a number of lattice points in F_M (6) which is fixed by our choice of some positive integer M. We arrange the data of the lattice points of F_M in a column. The result of the multiplication of the decomposition matrix by a column of data gives the Fourier coefficients in the expansion. The most valuable property of decomposition matrices is their independence of any particular data. Moreover, such matrices can be calculated once and then can be reused when some data on the same set of points of the fundamental domain F_M is considered.

2. A_n lattice and its refinement

Let $\mathbb{R}^n$ be the n-dimensional real Euclidean space endowed with the scalar product $\langle\ ,\ \rangle$. Let r_α be the *reflection* applied to the simple root α_j of A_n and

$$P_\alpha = \{\alpha_i \quad \mathbb{R}^n : \langle \alpha_i, \alpha_j \rangle = 0,\ i, j = 1, \dots, n\}$$

be the corresponding *reflecting hyperplane* in $\mathbb{R}^n$. Therefore, the reflection is defined by

$$r_j x = x - \frac{2\langle x, \alpha_j\rangle}{\langle \alpha_j, \alpha_j\rangle}\alpha_j, \quad \text{where} \quad j = 1, 2, \dots, n \quad \text{and} \quad x \quad \mathbb{R}^n. \tag{1}$$

A linear combination of the simple roots of A_n forms the *root system* $\Delta_{A_n} = \{\alpha_1, \dots, \alpha_n\}$ that represents a non-orthogonal basis in $\mathbb{R}^n$. The finite reflections applied to Δ_{A_n} generate a finite reflection group $\mathcal{W}_{A_n}$ (or *Weyl group*) of order $|\mathcal{W}_{A_n}| = (n+1)!$. The characteristic property for the reflections is the equality $r_j^2 = 1$. Every reflection r_j can be attached to the jth node (that stands for a simple root) of the corresponding Coxeter–Dynkin diagram (Fig. 1).

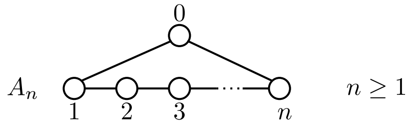


FIGURE 1. Extended Coxeter–Dynkin diagram for the Lie algebra of type A_n.

The set of ω_k $(k = 1, \dots, n)$ is called the *set of fundamental weights*. The α- and ω-bases are not orthogonal, but they are dual to each other in the following sense:

$$\frac{2\langle \alpha_j, \omega_k\rangle}{\langle \alpha_j, \alpha_j\rangle} = \delta_{jk}, \quad \text{where} \quad j, k \quad 1, \dots, n. \tag{2}$$

The link between α- and ω-bases is given by the *Cartan matrix* $\mathcal{C}$ and its inverse $\mathcal{C}^{-1}$:

$$\alpha_j = \sum_{k=1}^{n} \mathcal{C}_{jk}\omega_k, \quad \omega_j = \sum_{k=1}^{n} \left(\mathcal{C}_{jk}^{-1}\right)\alpha_k. \tag{3}$$

All linear combinations of the simple roots of A_n form the *root lattice* Q_{A_n} which has the following form:

$$Q_{A_n} = \left\{ \sum_{i=1}^{n} a_i \alpha_i \mid a_i \quad \mathbb{Z} \right\} \equiv \bigoplus_i \mathbb{Z}\alpha_i \equiv \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_n. \tag{4}$$

Thus, we can introduce the *weight lattice* P_{A_n} of A_n which is denoted as

$$P_{A_n} = \left\{ \sum_{j=1}^{n} b_j \omega_j \mid b_j \quad \mathbb{Z} \right\} \equiv \bigoplus_j \mathbb{Z}\omega_j \equiv \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n. \tag{5}$$

Let F be the fundamental region of P_{A_n}. We can refine the basic tile F to the smaller tiles F_M of the same shape. The number of lattice points of F_M inside F (including points on its boundary) is given as the function of M, where M stands for any positive integer [2]:

$$|F_M| = \binom{M+n}{n}. \tag{6}$$

If F_M is used to tile entire $\mathbb{R}^n$, then the points of F_M form a refined infinite lattice $L(F_M)$ with a density fixed by M. The greater the value of M, the finer the obtained lattice is. Geometrically it is the same lattice as $L(F)$, with distances scaled down by a factor of $1/M$. Finding all points of F_M amounts to finding all distinct $(n+1)$ non-negative integers $[s_0, s_1, s_2, \ldots, s_n]$ (called barycentric coordinates) that add up to M:

$$M = \sum_{i=0}^{n} s_i, \quad s_i \quad \mathbb{Z}^{\geq 0}, \; i \quad \{0, 1, \ldots, n\}. \tag{7}$$

Then the set of discrete points F_M is given by

$$F_M = \left\{ \frac{s_1}{M}\omega_1 + \cdots + \frac{s_n}{M}\omega_n \,\middle|\, [s_0, s_1, s_2, \ldots, s_n] \quad I_M \right\}, \tag{8}$$

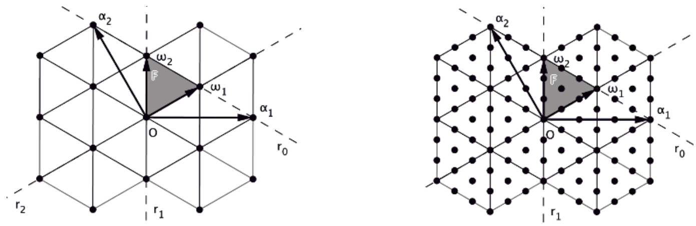
where the basis vectors ω_i, $i \quad \{1, 2, \ldots, n\}$ are the sides of simplex $F(A_n)$, and is labeled by the index set

$$I_M = \left\{ [s_0, s_1, \ldots, s_n] \quad \left(\mathbb{Z}^{\geq 0}\right)^{n+1} \,\middle|\, \sum_{i=0}^{n} s_i = M \right\}. \tag{9}$$

Example. Here we present just two special cases for the lower case A_2:

$$\begin{aligned} M = 1 \;:&\quad [1,0,0],\; [0,1,0],\; [0,0,1]\,, \\ M = 3 \;:&\quad [3,0,0],\; [0,3,0],\; [0,0,3],\; [1,1,1],\; [0,2,1], \\ &\quad [2,1,0],\; [1,0,2],\; [2,0,1],\; [1,2,0],\; [0,1,2]\,. \end{aligned}$$

Starting from any point of $\mathbb{R}^2$ we can generate the two lattices, namely $F_1(A_2)$ and $F\ (A_2)$, by application of repeated reflections as shown in Fig. 2.



$M = 1$ $M = 3$

FIGURE 2. Fragments of the infinite lattices $L(F_1(A_2))$ and $L(F_3(A_2))$. The mirrors $_1$, $_2$ are marked by the dashed lines. The points of the generated lattices are marked by black dots. The vertices of the basic tile $F(A_2) = F_1(A_2)$ are marked as 0, ω_1 and ω_2. The normal vectors to the reflection mirrors are denoted as α_1 and α_2. The shaded triangle corresponds to the basic tile $F(A_2)$.

3. The congruence classes of points of F_M

We introduce the congruence classes as an extension of the triality for A_n modules [5, 6]. This is a powerful tool in calculations of the decomposition matrices. The general idea is to consider a quotient of two components: the integral lattice of fundamental weights and the integral lattice of simple roots. Since the weight spaces of any finite dimensional highest weight representation are clearly contained in a set consisting of a highest weight of the representation (i.e. an element in an integral lattice of fundamental weights) and the integral lattice of simple roots, we can simply associate some element of the quotient with this representation.

Any point of the lattice $x \quad L(A_n)$ can be split into $n+1$ congruence classes by using the following rule:

$$x = \sum_{i=1}^{n} a_i \omega_i \quad K_k, \quad \text{where } \sum_{i=1}^{n} i \cdot a_i = k \mod (n+1). \tag{10}$$

Example. Consider A_2 case. Any point $x \quad L(A_2)$ can be split into three mutually congruence classes. The general rule is the following:

$$x = a_1\omega_1 + a_2\omega_2 \quad K_k, \quad \text{where } a_1 + 2a_2 = k \mod 3.$$

4. C- and S-functions

In this section we consider two special functions, C- and S- orbit functions, of a simple Lie algebra A_n. They are defined by means of a finite reflection group $\mathcal{W}$ [3, 4]. Both the S- and C-functions (multidimensional *cosine* and *sine* functions, respectively) are the linear combination of exponential expressions and the difference between them is the presence or absence of negative signs in front of the sums of the exponential terms. For the C-function all terms have a positive sign, hence, we call it a *symmetric orbit function.* Thus, we define a C-orbit function as

$$C_\lambda(x) = \sum_{\omega\in\mathcal{W}(\lambda)} e^{2\pi i\langle\omega(\lambda),x\rangle}, \tag{11}$$

where λ stands for the dominant weight and x stands for n parameters of the element of the Lie algebra A_n. All the elements of the same orbit are equidistant from the origin and x can take any value from $\mathbb{R}^n$. An S-function, also referred as *antisymmetric orbit function*, has the following form:

$$S_\lambda(x) = \sum_{\omega\in\mathcal{W}(\lambda)} (\det\omega)e^{2\pi i\langle\omega(\lambda),x\rangle}. \tag{12}$$

The number of terms in the expressions for the special functions does not exceed the order of the corresponding Weyl group. The summation proceeds over the whole orbit $\mathcal{W}$ and the number of the summands correspond to the size of the orbit $|\mathcal{W}(\lambda)|$.

4.1. Orthogonality of the orbit functions

The orthogonality of the C- and S-orbit functions for the simple Lie algebra A_n is defined by the following formulas [2, 3]:

$$\langle C_\lambda(x), C_{\lambda'}(x)\rangle = \sum_{j\in I_M} \frac{|\mathcal{W}|}{|\mathcal{W}_x|} C_\lambda(x_j)\overline{C_{\lambda'}(x_j)} = \det\mathcal{C}|\mathcal{W}|M^n|\mathcal{W}_\lambda|\delta_{\lambda,\lambda'}, \tag{13}$$

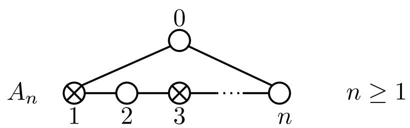
$$\langle S_\lambda(x), S_{\lambda'}(x)\rangle = \sum_{j\in I_M} S_\lambda(x_j)\overline{S_{\lambda'}(x_j)} = \det\mathcal{C}M^n\delta_{\lambda,\lambda'}, \quad x_j \quad F_M, \tag{14}$$

where $|\mathcal{W}_x|$ and $|\mathcal{W}_\lambda|$ are the orders of the stabilizers of elements x and λ, respectively [2].

Let us take the extended Dynkin diagram of the Lie algebra A_n (Fig. 4.1). Each node corresponds to the s_i coordinate, $i \quad \{0,1,\ldots,n\}$. Let's choose a point $x \quad F_M$ for any M and its barycentric coordinates $[s_0, s_1, \ldots, s_n]$ such as:

1. If $s_i > 0$ for all $i \quad \{0,1,\ldots,n\}$, then $|\mathcal{W}_x| = 1$,
2. If there is $s_i = 0$ for any $i \quad \{0,1,\ldots,n\}$, then the extended Dynkin diagram decomposes to several non-extended subdiagrams of A_l. The order of the stabilizer of x we calculated by the formula $|\mathcal{W}_x| = \prod_l |\mathcal{W}_l|$, where $\mathcal{W}_l$ is a Weyl group of A_l.

Example. Consider a point with the coordinates $[s_0, 0, s_2, 0, s_4, \ldots, s_n]$. The corresponding diagram is the following



This Dynkin diagram decomposes into two Dynkin diagrams of A_1 and A_{n-2}. Then

$$|\mathcal{W}_x| = |\mathcal{W}(A_1)| \cdot |\mathcal{W}(A_{n-2})| = 2 \cdot (n-1).$$

5. Products of orbit functions

C-orbit functions are called symmetric, because they are invariant with respect to the action of the Weyl group of A_n. Since the product also has to be invariant under the Weyl group it necessarily decomposes into a sum of several C-functions. The product of two S-functions is also invariant under the action of the Weyl group. Therefore, it also decomposes into a sum of several C-functions. The product of a C- and S-function is skew-invariant with respect to Weyl group and, therefore, it decomposes into a sum of several S-functions.

Example. Let one consider the group $SU(3)$. Then the product of C-functions has the following form:

$$\begin{aligned}
C_{a,0}(x) \cdot C_{0,b}(x) =& \ C_{a,b}(x) + 2C_{a-b,0}(x), \qquad b \le a,\\
C_{a,0}(x) \cdot C_{0,b}(x) =& \ C_{a,b}(x) + 2C_{0,b-a}(x), \qquad a \le b,\\
C_{a,b}(x) \cdot C_{c,0}(x) =& 2C_{a+c,b}(x) + 2C_{c-a,a+b}(x) + 2C_{a,b-c}(x), \qquad a \le c \le b,\\
C_{a,b}(x) \cdot S_{a,b}(x) =& C_{2a,2b}(x) - 2C_{b,a}(x), \qquad a \le b,\\
C_{c,0}(x) \cdot S_{a,b}(x) =& 2S_{a+c,b}(x) + 2S_{c-a,a+b}(x) - 2S_{a,b-c}(x), \qquad a \le c \le b.
\end{aligned}$$

6. Decomposition matrices

A decomposition matrix is a useful tool for calculating Fourier expansions of digital data sampled on the points of the given lattice fragment F_M [1]. Moreover, a technique of finding fast Fourier transforms also involves decomposition matrices [7].

The Fourier transform on F_M is an exact equality between some data function sampled on the points of F_M and the finite series of coefficients multiplied by orbit functions

$$F(x) = \sum_{\lambda} f_\lambda \Phi_\lambda(x)\,, \qquad x \quad F_M,\ \lambda \quad \Lambda_M, \tag{15}$$

where Φ_λ is any of the C- or S-functions defined in Section 4. The number of terms in the sum doesn't exceed the number of points in F_M. The orthogonality of orbit functions allows one to express the coefficients of the expansion as a product of the decomposition matrix $D^{[M]}$ multiplied by the column of values of the data sampled on the points of F_M

$$f_\lambda = \sum_{x \in F_M} D^{[M]}_{(\lambda)(x)} F(x), \tag{16}$$

where

$$D^{[M]}=\left(D^{[M]}_{(\lambda)(x)}\right)=\left(\frac{\frac{|\mathcal{W}|}{|\mathcal{W}_x|}\overline{\Phi_\lambda(x)}}{\det\mathcal{C}|\mathcal{W}|M^n|\mathcal{W}_\lambda|}\right). \tag{17}$$

The main property of a decomposition matrix is its independence of any data function. This matrix can be calculated only once and later can be used for different data sets sampled on the same set of points of F_M.

Example. Let us consider a function $f(x,y)=\ln\cos\sin(\pi(x+2y))$ (Fig. 3). We use the decomposition matrix $D^{[2]}$ for $M=2$ of the group $SU(3)$ to approximate the function $f(x,y)$. Therefore, we obtain the matrix that has the following form:

$$D^{[2]}=\begin{pmatrix}
-\frac{1}{144}-\frac{i}{48\sqrt{3}} & \frac{1}{24} & -\frac{1}{144}+\frac{i}{48\sqrt{3}} & -\frac{1}{48}+\frac{i}{16\sqrt{3}} & -\frac{1}{48}-\frac{i}{16\sqrt{3}} & \frac{1}{72}\\
\frac{1}{24} & -\frac{1}{24} & \frac{1}{24} & -\frac{1}{24} & -\frac{1}{24} & \frac{1}{24}\\
-\frac{1}{144}+\frac{i}{48\sqrt{3}} & \frac{1}{24} & -\frac{1}{144}-\frac{i}{48\sqrt{3}} & -\frac{1}{48}-\frac{i}{16\sqrt{3}} & -\frac{1}{48}+\frac{i}{16\sqrt{3}} & \frac{1}{72}\\
-\frac{1}{48}+\frac{i}{16\sqrt{3}} & -\frac{1}{24} & -\frac{1}{48}-\frac{i}{16\sqrt{3}} & \frac{1}{48}+\frac{i}{16\sqrt{3}} & \frac{1}{48}-\frac{i}{16\sqrt{3}} & \frac{1}{24}\\
-\frac{1}{48}-\frac{i}{16\sqrt{3}} & -\frac{1}{24} & -\frac{1}{48}+\frac{i}{16\sqrt{3}} & \frac{1}{48}-\frac{i}{16\sqrt{3}} & \frac{1}{48}+\frac{i}{16\sqrt{3}} & \frac{1}{24}\\
\frac{1}{72} & \frac{1}{24} & \frac{1}{72} & \frac{1}{24} & \frac{1}{24} & \frac{1}{72}
\end{pmatrix}.$$

Values of the function $f(x,y)$ on the lattice points are presented in the table below.

(x,y)	$(0,1)$	$(\frac{1}{2},\frac{1}{2})$	$(1,0)$	$(0,\frac{1}{2})$	$(\frac{1}{2},0)$	$(0,0)$
$f(x,y)$	0	$\ln\cos 1$	0	0	$\ln\cos 1$	0

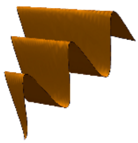


FIGURE 3. The function $f(x,y)=\ln\cos\sin(\pi(x+2y))$ plotted over the domain F of A_2.

Multiplying the matrix $D^{[2]}$ by the column of values of the function $f(x,y)$ we get the coefficients of the Fourier expansion of this function. Thus, the Fourier decomposition can be written explicitly as follows:

$$\begin{aligned}
f(x,y) \approx \frac{1}{12} \ln\cos(1)\Big(&- \sqrt{3}\sin\left(\frac{2}{3}\pi(2x+y)\right) - \sqrt{3}\sin\left(\frac{4}{3}\pi(2x+y)\right) \\
&+ \sqrt{3}\sin\left(\frac{2}{3}\pi(x-y)\right) + \sqrt{3}\sin\left(\frac{4}{3}\pi(x-y)\right) + \sqrt{3}\sin\left(\frac{2}{3}\pi(x+2y)\right) \\
&+ \sqrt{3}\sin\left(\frac{4}{3}\pi(x+2y)\right) - \cos\left(\frac{2}{3}\pi(x-y)\right) + \cos\left(\frac{4}{3}\pi(x-y)\right) \\
&- 2\cos(2\pi(x+y)) - \cos\left(\frac{2}{3}\pi(2x+y)\right) + \cos\left(\frac{4}{3}\pi(2x+y)\right) \\
&- \cos\left(\frac{2}{3}\pi(x+2y)\right) + \cos\left(\frac{4}{3}\pi(x+2y)\right) - 2\cos(2\pi x) - 2\cos(2\pi y) + 6\Big).
\end{aligned}$$

It is easy to verify that the decomposed one has the same values on the lattice points of F_M as the original $f(x,y)$. In Fig. 4 we present plots of the approximation function $f(x,y)$ for $M = 2$ and $M =$. The higher the value of M, the better approximation of $f(x,y)$ is.

$M = 2$ $M =$

FIGURE 4. Plots of the interpolating functions for $M = 2$ and $M = 4$ with number of points 6 and 15, respectively.

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The Super Orbit Challenge

Gijs M. Tuynman

Abstract. When using the generally adopted definition of a super unitary representation, there are lots of super Lie groups for which the regular representation is not super unitary. I propose a new definition of a super unitary representation for which all regular representations are super unitary. I then choose a particular super Lie group (of Heisenberg type) for which I provide a list of super unitary representations in my new sense, obtained by a heuristic super orbit method. The super orbit challenge is to find a well defined super orbit method that will provide (for a suitable category of super Lie groups) the full super-unitary dual and that reproduces the list of my super unitary representations (or explains why they should not appear).

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Keywords. Super unitary representation, super orbit method.

1. Introduction

In order to make this paper as easily accessible as possible, I will interpret a super Lie group as a super Harish-Chandra pair $(G_o, \mathfrak{g})$, even though I prefer to interpret them as a supermanifold G with a compatible group structure (in the sense of $\mathcal{A}$-manifolds [5]). In a super Harish-Chandra pair $(G_o, \mathfrak{g})$, $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a super Lie algebra (over $\mathbf{R}$) and G_o an ordinary Lie group acting on $\mathfrak{g}$ such that:

A1. The Lie algebra of G_o is (isomorphic to) $\mathfrak{g}_0$;
A2. The action of G_o preserves each $\mathfrak{g}_\alpha$ (the action is "even");
A3. The restriction of the G_o action to $\mathfrak{g}_0$ is (isomorphic to) the adjoint action of G_o on it Lie algebra.

The generally accepted definition of a super unitary representation of a super Lie group $(G_o, \mathfrak{g})$ is the one that can be found (among others) in [3, Def. 2, §2.3] and [1]. One defines a super Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle, \mathscr{S})$, as a graded Hilbert space

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$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ with scalar product $\langle \cdot, \cdot \rangle$ and super scalar product $\mathscr{S}$ (a graded symmetric non-degenerate sesquilinear form) satisfying the following conditions:

B1. $\langle \mathcal{H}_0, \mathcal{H}_1 \rangle = 0$;

B2. For all homogeneous $x, y \quad \mathcal{H}$ we have $\mathscr{S}(x, y) = i^{|x|} \cdot \langle x, y \rangle$.

With these ingredients a super unitary representation of $(G_o, \mathfrak{g})$ on the super Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle, \mathscr{S})$ then is a couple (ρ_o, τ) in which ρ_o is an ordinary unitary representation of G_o on the Hilbert space $\mathcal{H}$ and $\tau : \mathfrak{g} \to \mathrm{End}\big(C^\infty(\rho_o)\big)$ an even super Lie algebra representation of $\mathfrak{g}$ on $C^\infty(\rho_o)$, the space of smooth vectors for ρ_o defined by

$$C^\infty(\rho_o) = \{\, \psi \quad \mathcal{H} \mid g \mapsto \rho(g)\psi \text{ is a smooth map } G \to \mathcal{H} \,\},$$

satisfying the conditions:

C1. For each $g \quad G_o$ the map $\rho_o(g)$ preserves each $\mathcal{H}_\alpha$ (the representation is "even");

C2. For each $X \quad \mathfrak{g}_0$ (the Lie algebra of G_o!) the map $\tau(X)$ is the restriction of the infinitesimal generator of $\rho_o\big(\exp(tX)\big)$ to $C^\infty(\rho_o)$;

C3. For each $X \quad \mathfrak{g}_\alpha$ the map $\tau(X)$ is graded skew-symmetric with respect to $\mathscr{S}$;

C4. For all $g \quad G_o$ and all $X \quad \mathfrak{g}_1$ we have

$$\tau(g \cdot X) = \rho_o(g) \circ \tau(X) \circ \rho_o(g^{-1}),$$

where on the left we denote by $g \cdot X$ the action of G_o on $\mathfrak{g}$.

Unfortunately, already for the simplest super Lie group $\mathbf{R}^{0|1}$, the $0|1$-dimensional abelian super Lie group for which the super Harish-Chandra pair is $(\{e\}, \{0\} \oplus \mathbf{R})$, the (left-) regular representation is not super unitary in the above sense. The representation space is the space of (smooth) functions $C^\infty(\mathbf{R}^{0|1})$ of a single odd variable, i.e., isomorphic to $\mathbf{C}^2$ via $f(\xi) = a_0 + a_1\xi$ and the infinitesimal action is given by the operator ∂_ξ, i.e., by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. As $\mathbf{C}^2 = \mathbf{C} \oplus \mathbf{C}$ is $1|1$-dimensional, there is no possible choice for a super Hilbert space structure on $\mathbf{C}^2$ for which the regular representation is super unitary.

In [9],[1] I proposed to change the definition of a super Hilbert space to a triple $(\mathcal{H}, \langle \cdot, \cdot \rangle, \mathscr{S})$ by changing the condition B2 to

B'2. $\mathscr{S}$ is continuous with respect to the topology of $\mathcal{H}$ defined by $\langle \cdot, \cdot \rangle$.

But remember, $\mathscr{S}$ is just a non-degenerate graded symmetric sesquilinear form, not necessarily even nor homogeneous. Associated to this new definition of a super Hilbert space, I changed the definition of a super unitary representation on a super Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle, \mathscr{S})$ as a triple $(\rho_o, \mathcal{D}, \tau)$ in which ρ_o is an ordinary unitary representation of G_o on the Hilbert space $\mathcal{H}$ and $\tau : \mathfrak{g} \to \mathrm{End}\big(\mathcal{D}\big)$ an even super Lie algebra representation of $\mathfrak{g}$ on $\mathcal{D} \subset C^\infty(\rho_o) \subset \mathcal{H}$, a dense graded subspace of $\mathcal{H}$ contained in the set of smooth vectors of the unitary representation ρ_o, satisfying the conditions:

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[1] As [9] was too long for most journals, a shortened version without the sections on Berezin–Fourier decomposition will appear as [10].

C'1. for each g G_o the map $\rho_o(g)$ preserves each $\mathcal{H}_\alpha$ (the representation is "even");
C'2. For each X $\mathfrak{g}_0$ (the Lie algebra of G_o!) the map $\tau(X)$ is the restriction of the infinitesimal generator of $\rho_o\big(\exp(tX)\big)$ to $\mathcal{D}$;
C'3. For each X $\mathfrak{g}_\alpha$ the map $\tau(X)$ is graded skew-symmetric with respect to $\mathscr{S}$;
C'4. For all g G_o and all X $\mathfrak{g}_1$ we have

$$\tau(g \cdot X) = \rho_o(g) \circ \tau(X) \circ \rho_o(g^{-1});$$

C'5. $\mathcal{D} \subset \mathcal{H}$ is maximal with respect to the four conditions above.

Using these changed definitions, I was able to show that the left-regular representation of any connected super Lie group is super unitary in this new sense.[2] In particular for the simplest example of the $0|1$-dimensional super Lie group cited above, it suffices to take an odd super scalar product $\mathscr{S}$ instead of an even one as imposed by the standard definition.

Now I think that rendering all regular representations super unitary is sufficient reason to justify my change of the definition of a super unitary representation, but my initial motivation comes from a heuristic super version of the orbit method. In [8], I introduced the notion of a mixed symplectic form and I showed that coadjoint orbits of a super Lie group carry in a natural way such a mixed symplectic form. In [6] (see also [7]) I then showed that representations associated to orbits with a non-homogeneous symplectic form appear in the (Fourier–Berezin) decomposition of the regular representation of an explicit example of dimension $|$, justifying the introduction of non-homogeneous symplectic forms.[4] Now there seems to be a certain reluctance to accept the notion of non-even symplectic forms (see, for instance, [2]) and my "justifying" paper [6] has a serious drawback: half of the used procedure is heuristic and no (super) Hilbert spaces are mentioned.

I still have no satisfactory way to produce (by means of super geometric quantization of super symplectic manifolds with a non-even symplectic form) structures that might lead to super Hilbert spaces; for super Lie groups, I only have a systematic way to produce representations (essentially on spaces of smooth functions) associated to coadjoint orbits and polarizations, and then I have to invent by hand the (super) Hilbert space structure adapted to such a representation and I have to adapt by hand the dependence on odd parameters linked to the specific orbit. But now that I have a convenient notion of a super unitary representation, I will give, for a particular (Heisenberg like) super Lie group of dimension $3|3$, a list of super unitary representations in my new sense. I am convinced this will be the complete list of all inequivalent irreducible super unitary representations of this group, but (of course) I have no proof and I might be wrong. And then the challenge is to find

---

[2]A slightly less far going modification of the notion of a super Hilbert space and an associated notion of a super unitary representation is proposed in [4].

[3]The timeline of the official publications is different from the production timeline as can be seen from the arXiv dates.

[4]The same Fourier–Berezin decomposition technique was used in [2] to decompose the regular representation of the $0|$ -dimensional super Lie group described above.

a systematic way to obtain them via a well-defined orbit method. The interested reader will find another example in [6] (for which I have the same conviction) to test any super orbit method, although in that example no mention is made of any kind of notion of super unitary representation.

2. A super Lie group and a list of super unitary representations

As a super Lie group of dimension 3|3 our example G "is" $\mathbf{R}^{\,|\,}$ with three global even coordinates a, b, c and three global odd coordinates α, β, γ and group law given by the multiplication

$$(a, b, \alpha, \beta, c, \gamma) \cdot (\hat{a}, \hat{b}, \hat{\alpha}, \hat{\beta}, \hat{c}, \hat{\gamma}) = \big(a + \hat{a}, b + \hat{b}, \alpha + \hat{\alpha}, \beta + \hat{\beta},$$
$$c + \hat{c} + \tfrac{1}{2}(a\hat{b} - b\hat{a} - \alpha\hat{\beta} - \beta\hat{\alpha}), \gamma + \hat{\gamma} + \tfrac{1}{2}(a\hat{\beta} - \beta\hat{a} + b\hat{\alpha} - \alpha\hat{b})\big).$$

As a super Harish-Chandra pair $(G_o, \mathfrak{g})$ it is given by the standard Heisenberg group $G_o = \mathbf{R}$ of dimension 3 with group law

$$(a, b, c) \cdot (\hat{a}, \hat{b}, \hat{c}) = \big(a + \hat{a}, b + \hat{b}, c + \hat{c} + \tfrac{1}{2}(a\hat{b} - b\hat{a})\big).$$

The super Lie algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ of dimension 3|3 with three even basis vectors e_0, e_1, e_2 and three odd basis vectors f_0, f_1, f_2 is described by the commutators

$$[e_1, e_2] = e_0 = [f_1, f_2], \quad [e_1, f_2] = f_0 = [e_2, f_1],$$

all others either 0 or determined by graded skew-symmetry. It is a central extension of the abelian super group of dimension 2|2 by a 1|1-dimensional center; at the algebra level the center is generated by the vectors e_0, f_0. And finally the (adjoint) action of G_o on $\mathfrak{g}$ is given by

$$\begin{aligned} &(a, b, c) \cdot e_0 = e_0, \qquad &&(a, b, c) \cdot e_1 = e_1 - b\, e_0, \qquad &&(a, b, c) \cdot e_2 = e_2 + a\, e_0, \\ &(a, b, c) \cdot f_0 = f_0, \qquad &&(a, b, c) \cdot f_1 = f_1 + b\, f_0, \qquad &&(a, b, c) \cdot f_2 = f_2 + a\, f_0. \end{aligned}$$

Once we have the description of our super Lie group, we can provide our list of seven families of super unitary representations. However, instead of providing the unitary representation ρ_o of G_o and the infinitesimal representation τ, I will give the integrated version ρ, which is a bona fide representation of the full super group G. The unitary representation ρ_o is directly obtained by putting $\alpha = \beta = \gamma = 0$ in the expression for ρ, and τ is obtained by computing the derivatives of ρ with respect to the six variables $a, b, c, \alpha, \beta, \gamma$ at the point $(a, b, c, \alpha, \beta, \gamma) = \mathbf{0}$. For the third family this will be done explicitly.

Family 1. We start with a family of 1-dimensional representations depending on two real parameters k, ℓ and two odd parameters κ, λ. Our graded Hilbert space is given by $\mathcal{H} = \mathbf{C} \oplus \{0\}$ with scalar product and super scalar product $\mathscr{S}(\chi, \psi) = \langle \chi, \psi \rangle = \overline{\chi} \cdot \psi$. And then the representation $\rho^{(1)}_{k,\ell,\kappa,\lambda}$ is given by

$$\rho^{(1)}_{k,\ell,\kappa,\lambda}(a, b, c, \alpha, \beta, \gamma)\psi = \mathrm{e}^{i(ak + b\ell + \alpha\kappa + \beta\lambda)}\, \psi.$$

Family 2. For this family the Hilbert space is $\mathcal{H} = L^2(\mathbf{R}^2) \oplus \{0\}$ with its standard scalar product and super scalar product given by

$$\langle \chi, \psi \rangle = \mathscr{S}(\chi, \psi) = \int \overline{\chi(x,y)}\, \psi(x,y)\ \mathrm{d}x\, \mathrm{d}y.$$

On this Hilbert space we define a one-parameter family of representations $\rho_\kappa^{(2)}$ depending on a nonzero odd parameter κ by

$$\big(\rho_\kappa^{(2)}(a,b,c,\alpha,\beta,\gamma)\psi\big)(x,y) = \psi(x+b, y+a)\, \mathrm{e}^{i\alpha\kappa x}\, \mathrm{e}^{i\beta\kappa y}\, \mathrm{e}^{i(\gamma + \frac{1}{2}(\beta a + b\alpha))\kappa}.$$

Family 3. Here the graded Hilbert space is $\mathcal{H}$ is given by $\mathcal{H} = L^2(\mathbf{R}) \oplus L^2(\mathbf{R})$, which I interpret as functions of one even variable x and one odd variable ξ according to

$$(\psi_0, \psi_1) \quad L^2(\mathbf{R}) \oplus L^2(\mathbf{R}) \ \cong\ \psi(x,\xi) = \psi_0(x) + \xi\psi_1(x).$$

The scalar product $\langle \cdot, \cdot \rangle$ and the (odd) super scalar product are given by

$$\langle \chi, \psi \rangle = \int \overline{\chi_0(x)}\, \psi_0(x) + \overline{\chi_1(x)}\, \psi_1(x)\ \mathrm{d}x,$$
$$\mathscr{S}(\chi, \psi) = \int \overline{\chi_0(x)}\, \psi_1(x) + \overline{\chi_1(x)}\, \psi_0(x)\ \mathrm{d}x.$$

On this Hilbert space we define a one-parameter family of representations $\rho_k^{(\ )}$ depending on a nonzero real parameter k by

$$\big(\rho_k^{(\ )}(a,b,c,\alpha,\beta,\gamma)\psi\big)(x,\xi) = \psi(x+ka, \xi - k\alpha)\, \mathrm{e}^{ibx}\, \mathrm{e}^{-i\beta\xi}\, \mathrm{e}^{ik(c + \frac{1}{2}(ab - \alpha\beta))}.$$

This means that the unitary representation ρ_o is given by

$$\big(\rho_o(a,b,c)\psi\big)(x,\xi) = \psi(x+ka, \xi)\, \mathrm{e}^{ixb}\, \mathrm{e}^{i(c+\frac{1}{2}ab)k}$$

and the super Lie algebra representation is given by

$$\tau(e_0)\psi = ik\,\psi, \qquad \tau(e_1)\psi = k\,\frac{\partial\psi}{\partial x}, \qquad \tau(e_2)\psi = ix\,\psi,$$
$$\tau(f_0)\psi = 0, \qquad \tau(f_1)\psi = -k\,\frac{\partial\psi}{\partial\xi}, \qquad \tau(f_2)\psi = -i\xi\,\psi.$$

Family 4. For this family the Hilbert space is the same as for the third family. On it we define a two-parameter family of representations $\rho_{k,\kappa}^{(4)}$ depending on a real parameter k and a nonzero odd parameter κ by

$$\big(\rho_{k,\kappa}^{(4)}(a,b,c,\alpha,\beta,\gamma)\psi\big)(x,\xi) = \psi(x+a, \xi-\alpha)\, \mathrm{e}^{ib(k+\xi\kappa)}\, \mathrm{e}^{i\beta\kappa x}\, \mathrm{e}^{i(\gamma+\frac{1}{2}(\beta a - b\alpha))\kappa}.$$

Family 5. For this family the Hilbert space is again the same as for the third family. On it we define a two-parameter family of representations $\rho_{k,\kappa}^{(5)}$ depending on a nonzero real parameter k and a nonzero odd parameter κ by

$$\begin{aligned}\big(\rho_{k,\kappa}^{(5)}&(a,b,c,\alpha,\beta,\gamma)\psi\big)(x,\xi)\\ &= \psi(x+a,\xi-\alpha)\, \mathrm{e}^{ib(xk+\xi\kappa)}\, \mathrm{e}^{i\beta(x\kappa - \xi k)} \mathrm{e}^{i(\gamma+\frac{1}{2}(\beta a - b\alpha))\kappa}\, \mathrm{e}^{i(c+\frac{1}{2}(ab+\beta\alpha))k}.\end{aligned}$$

Family 6. Here the graded Hilbert space is $\mathcal{H} = \mathbf{C}^2 \oplus \mathbf{C}^2$, which I interpret as functions of two odd variables ξ and η according to

$$\big((\psi_0, \psi_{12}) \oplus (\psi_1, \psi_2)\big) \quad \mathbf{C}^2 \oplus \mathbf{C}^2 \;\cong\; \psi(\xi, \eta) = \psi_0 + \xi\,\psi_1 + \eta\,\psi_2 + \xi\eta\,\psi_{12}.$$

The standard scalar product $\langle \cdot, \cdot \rangle$ and the super scalar product $\mathscr{S}$ are given by

$$\begin{aligned}\langle \chi, \psi \rangle &= \overline{\chi_0}\,\psi_0 + \overline{\chi_{12}}\,\psi_{12} + \overline{\chi_1}\,\psi_1 + \overline{\chi_2}\,\psi_2,\\ \mathscr{S}(\chi, \psi) &= \overline{\chi_0}\,\psi_{12} + \overline{\chi_{12}}\,\psi_0 + \overline{\chi_1}\,\psi_2 - \overline{\chi_2}\,\psi_1.\end{aligned}$$

On this Hilbert space we define a three-parameter family of representations $\rho^{(6)}_{k,\ell,\kappa}$ depending on two real parameters k, ℓ and a nonzero odd parameter κ by

$$\begin{aligned}&\big(\rho^{(6)}_{k,\ell,\kappa}(a, b, c, \alpha, \beta, \gamma)\psi\big)(\xi, \eta)\\ &\qquad\qquad = \psi(\xi - \beta, \eta - \alpha)\, \mathrm{e}^{ia(\xi\kappa + k)}\, \mathrm{e}^{ib(\eta\kappa + \ell)}\, \mathrm{e}^{i(\gamma - \frac{1}{2}(\beta a + b\alpha))\kappa}.\end{aligned}$$

Family 7. For this family the Hilbert space is given by $\mathcal{H} = L^2(\mathbf{R})^2 \oplus L^2(\mathbf{R})^2$, which I interpret as functions of one even variable x and two odd variables ξ, η according to

$$\begin{aligned}&\big((\psi_0, \psi_{12}) \oplus (\psi_1, \psi_2)\big) \quad L^2(\mathbf{R})^2 \oplus L^2(\mathbf{R})^2\\ &\qquad\qquad \cong\; \psi(x, \xi, \eta) = \psi_0(x) + \xi\,\psi_1(x) + \eta\,\psi_2(x) + \xi\eta\,\psi_{12}(x).\end{aligned}$$

The scalar product $\langle \cdot, \cdot \rangle$ and the super scalar product $\mathscr{S}$ are given by

$$\begin{aligned}\langle \chi, \psi \rangle &= \int \overline{\chi_0(x)}\,\psi_0(x) + \overline{\chi_{12}(x)}\,\psi_{12}(x) + \overline{\chi_1}(x)\,\psi_1(x) + \overline{\chi_2(x)}\,\psi_2(x)\;\mathrm{d}x,\\ \mathscr{S}(\chi, \psi) &= \int \overline{\chi_0(x)}\,\psi_{12}(x) + \overline{\chi_{12}(x)}\,\psi_0(x) + \overline{\chi_1(x)}\,\psi_2(x) - \overline{\chi_2(x)}\,\psi_1(x)\;\mathrm{d}x.\end{aligned}$$

On this Hilbert space we define a three-parameter family of representations $\rho^{(7)}_{k,p,\kappa}$ depending on two nonzero real parameters k, p and a nonzero odd parameter κ by

$$\begin{aligned}&\big(\rho^{(7)}_{k,p,\kappa}(a, b, c, \alpha, \beta, \gamma)\psi\big)(x, \xi, \eta) = \psi(x + a - pb, \xi - \alpha, \eta - \beta)\\ &\qquad\qquad \times \mathrm{e}^{ib(xk + \xi\kappa + p\eta\kappa)}\, \mathrm{e}^{i\beta(x\kappa - \xi k)}\, \mathrm{e}^{i(\gamma - pb\beta + \frac{1}{2}(\beta a - b\alpha))\kappa}\, \mathrm{e}^{i(c + \frac{1}{2}(ab + \beta\alpha - pb^2))k}.\end{aligned}$$

3. Concluding remarks

• All (super) Hilbert spaces are interpreted as spaces of functions on super spaces of the form $\mathbf{R}^{p|q}$, or more precisely as spaces that we can interpret as $\mathcal{H} = C^\infty\big(\mathbf{R}^{0|q}; L^2(\mathbf{R}^p)\big)$, i.e., (smooth) functions of q odd variables with values in the space of square integrable functions of p real variables. It then turns out that in all cases the super scalar product $\mathscr{S}(\chi, \psi)$ is realized as the (translation invariant) Berezin–Lebesgue integral $\int_{\mathbf{R}^{p|q}} \overline{\chi(m)}\,\psi(m)\;\mathrm{d}m$.

• All (infinitesimal) representations τ act by differentiation or multiplication, and as such these operators act on the space of smooth functions $C^\infty(\mathbf{R}^{p|q})$. In all cases the unmentioned dense subspace $\mathcal{D}$ then is given by

$$\mathcal{D} = \left\{ \psi \quad C^\infty(\mathbf{R}^{p|q}) \mid \forall k \quad \mathbf{N}\ \forall X_1, \dots, X_k \quad \mathfrak{g} : \tau(X_1) \circ \cdots \circ \tau(X_k)\psi \quad \mathcal{H} \right\} .$$

• In my way of thinking, the first family is associated to (coadjoint) orbits of dimension 0|0, the third family is associated to orbits of dimension 2|2 with an even symplectic form, the families 2,   and 6 are associated to orbits of dimension 2|2 with an odd symplectic form, and the families 5 and 7 are associated to orbits of dimension 2|2 with a non-homogeneous symplectic form. The seventh family is atypical as it is obtained by a polarization that is not of "maximal" dimension. For $p = 0$ we recover (apart from the term ∂_η in $\tau(f_2)$) the fifth family with the additional variable η.

• In most of the families of representations we have required some of the parameters to be nonzero. Not because these representations do not exist when they take the value zero, but because in those cases the representation will certainly not be irreducible.

• The attentive reader will have noticed that I have not been completely honest, as the families 4–7 do not fit my description of a super unitary representation. In particular ρ_o is not an ordinary unitary representation of the ordinary Lie group G_o, due to the presence of the (supposedly nonzero) odd parameter κ. On the other hand, apart from the fact that some of the parameters are odd, all these representations definitely have a "unitary" look, especially when one realizes that the super scalar product is defined by integration of the product $\overline{\chi}\,\psi$ with respect to a translation invariant "measure." As moreover all these families are obtained in the same way, I am sorely tempted to want to enlarge the definition of a super unitary representation even more in order to include all these families. Unfortunately, I have not (as yet) a satisfactory way to do so.

• And last but not least: all feedback will be appreciated.

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Weighted generalization of the Szegö kernel and how it can be used to prove general theorems of complex analysis

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Abstract. We consider weighted generalization of the Szegö kernel. We show which conditions must a weight of integration satisfy in order for weighted Szegö kernel to exist. Then we show some properties of weighted Szegö kernel, including a direct formula for particular cases. At the end, we show how weighted Szegö kernel can be used to prove general theorems of complex analysis.

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1. Definitions and admissible weights

Let $\Omega \subset \mathbb{C}^N$ be a bounded domain with a boundary of class C^2. For $\mu : \partial\Omega \to \mathbb{R}$ measurable and almost everywhere greater than 0 (which we will call a **weight**) by $L^2(\partial\Omega, \mu)$ we will denote a set of functions $f : \partial\Omega \to \mathbb{C}$, square-integrable in the sense

$$\| f \|_\mu^2 := \int_{\partial\Omega} |f(w)|^2 \mu(w) \mathrm{d}S < \infty, \tag{1}$$

where the integral is understood as an integral of a scalar field with surface measure. The set $L^2(\partial\Omega, \mu)$ with an inner product given by

$$\langle f|g\rangle_\mu := \int_{\partial\Omega} \overline{f(w)} g(w)\mu(w) \mathrm{d}S \tag{2}$$

is a Hilbert space. Now let us consider the space $A(\Omega)$ of continuous functions $f : \overline{\Omega} \to \mathbb{C}$, such that $f_{|\Omega}$ is holomorphic. Let us denote

$$B(\Omega, \mu) := \left\{ f_{|\partial\Omega} : f \quad A(\Omega) \right\} \cap L^2(\partial\Omega, \mu).$$

By $L^2H(\partial\Omega,\mu)$ we will understand the closure of $B(\Omega,\mu)$ in $L^2(\partial\Omega,\mu)$ topology.

We will name space $L^2H(\partial\Omega,\mu)$ a **weighted Szegö space**. For $\mu \equiv 1$ it is a classical Szegö space.

Of course, $L^2H(\partial\Omega,\mu)$ can change as a set with a change of μ. However,

Theorem 1. *If μ_1, μ_2 are weights and there exist $m, M > 0$, such that*

$$m\mu_1(z) \le \mu_2(z) \le M\mu_1(z) \quad a.e., \tag{3}$$

then $L^2H(\partial\Omega,\mu_1) = L^2H(\partial\Omega,\mu_2)$ as a set.

In particular, if $0 < m < \mu < M < \infty$, then $L^2H(\partial\Omega,\mu) = L^2H(\partial\Omega,1)$ as a set.

For the proof look into [3].

Each element of $L^2H(\partial\Omega,1)$ has a unique holomorphic prolongation to Ω (see [1] for more details), so it is also true for any element from $B(\Omega,\mu)$, because $B(\Omega,\mu) \subset L^2H(\partial\Omega,1)$ for any μ. We will denote the set of all such prolongations by $\tilde{B}(\Omega,\mu)$ (where $\tilde{B}(\Omega,\mu) \subset A(\Omega)$). However, a good question to ask is how to find a holomorphic prolongation of functions from $L^2H(\partial\Omega,\mu) \setminus B(\Omega,\mu)$ for an arbitrary μ? We will answer this question in a moment.

We will use the same symbol for a function and its prolongation, which should not cause confusion.

Let μ be a weight with the following property:

(**CB**) For any compact set $X \subset \Omega$ there exists $C_X > 0$, such that for any $f \quad \tilde{B}(\Omega,\mu)$ and $z \quad X$

$$|f(z)| \le C_X \parallel f \parallel_\mu .$$

Then for functions from $L^2H(\partial\Omega,\mu) \setminus B(\Omega,\mu)$ we can define their prolongation to Ω in the following way:

Let f_n be a sequence of functions from $\tilde{B}(\Omega,\mu)$. Let $f \quad L^2H(\partial\Omega,\mu)$ be the limit of this sequence. Since by (CB) the sequence of functions $(f_{n|\Omega})$ fulfills the Cauchy condition locally uniformly on Ω, the function

$$f(z) := \lim_{n\to\infty} f_n(z), \quad z \quad \Omega$$

is well defined and holomorphic on Ω.

From now on, if μ fulfills (CB), we will interpret $L^2H(\partial\Omega,\mu)$ as a set of functions on $\overline{\Omega}$.

Let μ be a weight satisfying (CB). A function (if it exists) $S_\mu : \Omega \times \overline{\Omega} \to \mathbb{C}$, such that for any $z \quad \Omega$, $\overline{S_\mu(z,\cdot)} \quad L^2H(\partial\Omega,\mu)$ and for any $f \quad L^2H(\partial\Omega,\mu)$ (reproducing property)

$$f(z) = \left\langle \overline{S_\mu(z,\cdot)} | f(\cdot) \right\rangle_\mu, \tag{4}$$

will be called **Szegö kernel** of $L^2H(\partial\Omega,\mu)$.

It is true (as for any reproducing kernel Hilbert space) that if S_μ and S'_μ are Szegö kernels of the same space, then $S_\mu = S'_\mu$ and if the Szegö kernel exists, then it is given uniquely by a formula

$$S_\mu(z,w) = \sum_{i\in I} \varphi_i(z)\overline{\varphi_i(w)}, \tag{5}$$

where $\{\varphi_i\}_{i\in I}$ is an arbitrary complete orthonormal system of $L^2H(\partial\Omega,\mu)$.

S_μ is real analytic on $\Omega\times\Omega$ and by the Hartogs theorem on separate analyticity holomorphic with respect to the first n variables and antiholomorphic with respect to the last n variables.

It is a natural question to ask, which conditions must μ satisfy in order for $L^2H(\partial\Omega,\mu)$ to be a reproducing kernel Hilbert space.

Definition 2. We will say that a weight μ is **Szegö admissible** (S-admissible for short) if there exists Szegö kernel of $L^2H(\partial\Omega,\mu)$ space.

Theorem 3. *μ is an S-admissible weight if and only if the condition (CB) is satisfied.*

Proof. $\Longrightarrow$ comes directly from the definition.
$\Longleftarrow$ (CB) means that functionals of evaluation, i.e., functionals

$$\tilde{E}_z : \tilde{B}(\Omega,\mu) \ni f \mapsto f(z) \quad \mathbb{C}$$

are continuous. Since $B(\Omega,\mu)$ is dense in $L^2H(\partial\Omega,\mu)$ we can prolong $\tilde{E}_z$ to the functional $E_z \quad L^2H(\partial\Omega,\mu)^*$ with the same majoring constant C_X for any $z \quad \Omega$. By Riesz representation theorem, it means that for any E_z, where $z \quad \Omega$, there exists $e_z \quad L^2H(\partial\Omega,\mu)$, such that for any $f \quad L^2H(\partial\Omega,\mu)$

$$f(z) = \langle e_z | f\rangle$$

and the function

$$\overline{e_z(w)}, \quad (z,w) \quad \Omega\times\overline{\Omega}$$

is the Szegö kernel of $L^2H(\partial\Omega,\mu)$. □

Theorem 4. *Let μ be a weight on a bounded domain Ω with $\partial\Omega$ of class C^2, such that*

$$\int_{\partial\Omega} \frac{1}{\mu(w)}\mathrm{d}S < \infty. \tag{6}$$

Then μ is an S-admissible weight.

In order to prove the theorem we are going to use the following lemma:

Lemma 5. *Let Ω_1, Ω_2 be bounded domains with C^2-smooth boundaries, such that $\Omega_1 \subset \Omega_2$. Then there exists $C > 0$, such that for any $f \quad L^2H(\partial\Omega_2) = L^2H(\partial\Omega_2, 1)$ we have*

$$\int_{\partial\Omega_1} |f(w)|\mathrm{d}S \le C \int_{\partial\Omega_2} |f(w)|\mathrm{d}S$$

It is a particular case of Lemma 2.1 from [2], which was proven for $p > 1$. It remains true, however, for $p = 1$, since authors of [2] follow the proof of Theorem 1 from [4] and in the case of $p = 1$ we just need to change $f(y)\mathrm{d}\sigma(y)$ to finite Borel measure on $\partial\Omega$.

Moreover, since Ω is a bounded domain with $\partial\Omega$ of class C^2, therefore $\partial\Omega$ has finite measure and $f \quad L^2(\partial\Omega, \mu)$ implies that $f \quad L^1(\partial\Omega, \mu)$.

Proof. Let $z_0 \quad \Omega$ and let r be sufficiently small for

$$K_0 := K(z_0, 2r) := \{w \quad \mathbb{C}^N : |z_0 - w| < 2r\}$$

to lie with its boundary in Ω. Then by mean value theorem for harmonic functions we have for $f \quad \tilde{B}(\Omega, \mu)$, $z \quad K(z_0, r)$ and $K := K(z, r)$

$$|f(z)| = C_1 \left| \int_{\partial K} f(w)\mathrm{d}S \right| \le C_1 \int_{\partial K} |f(w)|\mathrm{d}S,$$

where $\frac{1}{C_1}$ is a measure of ∂K. By Lemma 5, we have

$$\int_{\partial K} |f(w)|\mathrm{d}S \le C_0 \int_{\partial K_0} |f(w)|\mathrm{d}S \le C_0 C_2 \int_{\partial\Omega} |f(w)|\mathrm{d}S.$$

By Schwarz inequality,

$$\begin{aligned} \int_{\partial\Omega} |f(w)|\mathrm{d}S &= \int_{\partial\Omega} |f(w)| \frac{\sqrt{\mu(w)}}{\sqrt{\mu(w)}}\mathrm{d}S \\ &\le \sqrt{\int_{\partial\Omega} |f(w)|^2 \mu(w)\mathrm{d}S} \sqrt{\int_{\partial\Omega} \frac{1}{\mu(w)}\mathrm{d}S}. \end{aligned}$$

Finally,

$$\begin{aligned} |f(z)| &\le C_0 C_1 C_2 \sqrt{\int_{\partial\Omega} \frac{1}{\mu(w)}\mathrm{d}S} \sqrt{\int_{\partial\Omega} |f(w)|^2 \mu(w)\mathrm{d}S} \\ &\le C_0 C_1 C_2 C \quad \| f \|_\mu \le C \| f \|_\mu, \end{aligned}$$

where C does not depend on $z \quad K(z_0, r)$. Hence μ satisfies (CB). □

Corollary 6. *If Ω is a bounded domain with a boundary of class C^2, then a weight μ defined on $\partial\Omega$ such that $\mu(z) \ge c > 0$ is an S-admissible weight.*

2. Properties of weighted Szegö kernel

Theorem 7. *Let Ω be a bounded domain with C^2-smooth boundary and μ be an S-admissible weight on $\partial\Omega$. Let*

$$L(\partial\Omega, \mu) := \int_{\partial\Omega} \mu(w)\mathrm{d}S. \tag{7}$$

Then for $z \quad \Omega$,

$$S_\mu(z,z) \geq \frac{1}{L(\partial\Omega,\mu)}.$$

In particular, if $L(\partial\Omega,\mu) < \infty$, *then* $S_\mu(z,z) > 0$.

Proof. If $L(\partial\Omega,\mu) < \infty$, then $1 \quad L^2H(\partial\Omega,\mu)$ and

$$\begin{aligned} 1 &= \int_{\partial\Omega} S_\mu(z,w)\mu(w)\mathrm{d}S \\ &\leq \sqrt{\int_{\partial\Omega} |S_\mu(z,w)|^2\mu(w)\mathrm{d}S}\sqrt{\int_{\partial\Omega} \mu(w)\mathrm{d}S} = \sqrt{S_\mu(z,z)}\sqrt{L(\partial\Omega,\mu)}, \end{aligned}$$

by reproducing property and Schwarz inequality. Taking square of both sides and dividing by $L(\partial\Omega,\mu)$ we obtain the thesis of the theorem.

If $L(\partial\Omega,\mu) = \infty$, then $\frac{1}{L(\partial\Omega,\mu)} = 0$ and $S_\mu(z,z) \geq 0$ is always fulfilled. □

Theorem 8. *Let* Ω *be a bounded domain with a boundary of class* C^2. *If* $f \quad H(\overline{\Omega})$ *is a function such that* $f(z) \neq 0$ *for any* $z \quad \overline{\Omega}$, *then* $\mu(z) := |f(z)|^2$ *is an* S*-admissible weight on* $\partial\Omega$, $L^2H(\partial\Omega,\mu) = L^2H(\partial\Omega,1)$ *as a set and the Szegö kernel* S_μ *of* $L^2H(\partial\Omega,\mu)$ *is equal to*

$$S_\mu(z,w) = \frac{1}{f(z)\overline{f(w)}} S_1(z,w), \tag{8}$$

where S_1 *is the Szeg¨ kernel of* $L^2H(\partial\Omega,1)$.

Proof. Because f is a continuous function on a compact set and $f(z) \neq 0$, $|f(z)| > c > 0$, on $\partial\Omega$, which means that $\mu(z) = |f(z)|^2$ is an S-admissible weight by Corollary 6. Since f is a continuous function on a compact set, we also have $|f(z)| < C < \infty$, so $L^2H(\partial\Omega,1)$ and $L^2H(\partial\Omega,\mu)$ are equal as sets, by Theorem 1.

Now let $\{\varphi_j\}_{j\in J}$ be a complete orthonormal system of $L^2H(\partial\Omega,1)$. Then $\{\psi_j\}_{j\in J}$ defined in the following way:

$$\psi_j(z) = \frac{1}{f(z)}\varphi_j(z)$$

is an element of $L^2H(\partial\Omega,\mu)$, because $L^2H(\partial\Omega,\mu) = L^2H(\partial\Omega,1)$ as a set. $\{\psi_j\}_{j\in J}$ is a complete orthonormal system in $L^2H(\partial\Omega,\mu)$, because

$$\begin{aligned} \langle\psi_j,\psi_k\rangle_\mu &= \int_{\partial\Omega} \frac{1}{\overline{f(z)}}\overline{\varphi_j(z)}\frac{1}{f(z)}\varphi_k(z)|f(z)|^2\mathrm{d}S \\ &= \int_{\partial\Omega} \overline{\varphi_j(z)}\varphi_k(z)\mathrm{d}S = \langle\varphi_j,\varphi_k\rangle_1. \end{aligned}$$

Finally

$$S_\mu(z,w) = \sum_j \frac{1}{f(z)}\varphi_j(z)\frac{1}{\overline{f(w)}}\varphi_j(w)$$
$$= \frac{1}{f(z)\overline{f(w)}}\sum_j \varphi_j(z)\overline{\varphi_j(w)} = \frac{1}{f(z)\overline{f(w)}}S_1(z,w). \qquad \square$$

3. How weighted Szegö kernel can be used to prove general theorems of complex analysis

Maximum modulus principle allows us to show that if two holomorphic functions are equal on a boundary of some bounded domain, then they are equal on the whole domain. In this section we will use the concept of weighted Szegö kernel to prove more general theorem:

Theorem 9. *Let Ω be a bounded domain with a boundary of class C^2. Let $f, g : \overline{\Omega} \to \mathbb{C}$, holomorphic on $\overline{\Omega}$ be functions such that $|f(z)| = |g(z)|$ on $\partial\Omega$ and $f(z), g(z) \neq 0$ for $z \quad \overline{\Omega}$. Then $|f(z)| = |g(z)|$ for $z \quad \overline{\Omega}$.*

Assumption that $f(z), g(z) \neq 0$ for $z \quad \overline{\Omega}$ is necessary, because e.g. functions z^k and z^l for $k \neq l$ have the same modulus on $\partial K(0,1)$, but their modulus is not the same on whole $K(0,1) := \{z \quad \mathbb{C} : |z| < 1\}$.

Proof. By Theorem 8 Szegö kernel of $L^2H(\partial\Omega,\mu)$ for $\mu = |f|^2$ is equal to

$$S_\mu(z,z) = \frac{1}{|f(z)|^2}S_1(z,z)$$

and at once

$$S_\mu(z,z) = \frac{1}{|g(z)|^2}S_1(z,z),$$

where S_1 is Szegö kernel of $L^2H(\partial\Omega,1)$. So we have

$$\frac{1}{|f(z)|^2}S_1(z,z) = \frac{1}{|g(z)|^2}S_1(z,z).$$

$S_1(z,z)$ is given for any $z \quad \Omega$, so that equality must be true on whole Ω. By Theorem 7, $S_1(z,z) > 0$, so we can divide both sides of the equation by $S_1(z,z)$ to get $|f(z)| = |g(z)|$ on whole Ω. $\square$

Corollary 10. *Let Ω be a bounded domain with a boundary of class C^2. If $f : \overline{\Omega} \to \mathbb{C}$ is holomorphic on $\overline{\Omega}$ and $|f(z)| = c$ for $z \quad \partial\Omega$ and $f(z) \neq 0$ for $z \quad \Omega$, then f is constant on $\overline{\Omega}$.*

Proof. Function g equal to c on whole $\overline{\Omega}$ satisfies assumptions of the corollary. By theorem 9, $|f(z)| = |g(z)|$ for any $z \quad \Omega$, so $|f(z)| = c$ on whole Ω. By Riemann–Cauchy equations, if a holomorphic function f has constant modulus on some open domain, then it is constant on the whole domain. $\square$

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Part VI

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Amenability, flatness and measure algebras

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My aim is to present some main ideas of the branch of the theory of algebras in functional analysis which is called homology theory of these algebras. I think, a good way to speak about this theory is to present some of its comparatively sound results with fine interplay of algebra and analysis, to formulate it, explain its ingredients and then to show how such a result is proved. Since we choose a really synthetic result, we have to discuss many basic things of the area.

Main Theorem. *Let G be continuous (= non-discrete) locally compact group. Then its Banach measure algebra $M(G)$ is not amenable.*

Combining this theorem with a known result of Johnson, which will be formulated later, we obtain

Corollary. *Let G be an arbitrary locally compact group. Then its (Banach) measure algebra $M(G)$ is amenable if and only if G is discrete and amenable in the group-theoretic sense.*

Recall that the **measure algebra** $M(G)$ consists of all finite complex regular Borel measures on G with the norm "variation". The multiplication, called **convolution**, is as follows: for $\mu, \nu \quad M(G)$ and a Borel subset $E \subseteq G$ we put

$$\mu * \nu(E) := \int_G \int_G \chi_E(st) d\mu(s) d\nu(t) = \int_G \nu(s^{-1}E) d\mu(s),$$

where χ denotes the characteristic function of a subset.

This algebra has an important two-sided ideal $M_c(G)$, consisting of the so-called continuous measures, that is measures, equal to zero in one-point sets. As a subspace in $M(G), M_c(G)$ has a Banach complement, which is the closure of the linear span of Dirac measures in different points. This fact will have a crucial role in our proof.

As to the **amenability**, it has a lot of equivalent definitions, but two approaches are basic. One of them has appeared in the West, and it is based on the notion of **derivation**, another one in the East, and it is based on the notion of a **flat module**. It is the latter approach that has led to a proof of the formulated theorem.

Let A be a Banach algebra, always supposed to have an identity, say e. A Banach space X is called **left Banach A-module, right Banach A-module or Banach A-bimodule**, if it is a module of a respective class in algebraic sense, and the outer multiplication(s) (denoted by the dot "$\cdot$") are continuous.

There are obviously defined notions of a Banach submodule of a given Banach module of every type, as well as of a quotient Banach module. An outstanding role in our proof will belong to the right quotient $M(G)$-module $M(G)/M_c(G)$, where the algebra and its ideal both are considered as right Banach $M(G)$-modules. The class of left, right A-modules and A-bimodules will be denoted by A-mod, mod-A and A-mod-A, respectively. Class of Banach spaces, that is $\boldsymbol{C}$-modules of every type, will be denoted by $\boldsymbol{Ban}$.

Speaking about operators between Banach spaces, we always suppose that they are bounded. For given Banach spaces E and F we shall denote, as usually, by $\mathcal{B}(E,F)$ the Banach space of all operators from E into F with the operator norm. It is easy to check that, if X and Y are left Banach A-modules, then $\mathcal{B}(X,Y)$ is a Banach A-bimodule with outer multiplication well defined by $(a\cdot\varphi)(x) = a\cdot(\varphi(x))$ and $(\varphi\cdot a)(x) = \varphi(a\cdot x)$.

Suppose we have a Banach A-bimodule X.

Definition 1. Let X A-mod-A. An operator $D : A \to X$ is called **derivation of the algebra A with values in X**, if it satisfies the "Leibnitz identity" $D(ab) = D(a)\cdot b + a\cdot D(b)$.

The question about the structure of derivations is very important for many reasons. For example, derivations are intimately connected with automorphisms of Banach algebras, and the latter, as physicists claim, have a physical sense.

Every x X gives rise to the derivation $D_x : a \mapsto a\cdot x - x\cdot a$. Such derivations are called **inner**. A typical question: for given A, X, is it true that an arbitrary derivation of A with values in X is inner?

The closed subspace in the space $\mathcal{B}(A,X)$, consisting of derivations, is denoted by $Z^1(A,X)$, and its (not necessarily closed) subspace, consisting of inner derivations, by $B^1(A,X)$. The quotient linear space $Z^1(A,X)/B^1(A,X)$ is denoted by $\mathcal{H}^1(A,X)$, and solemnly called **1-dimensional, cohomology group of A with coefficients in X**.

A natural question arises: what is the structure of such an algebra that all its derivations with arbitrary coefficients are inner, that is $\mathcal{H}^1(A,X) = 0$ for all X? Such a condition happened to be extremely rigid; we omit respective details. But Barry Johnson, a famous English mathematician, suggested the class of algebras that has, so to say, an optimal size; these are our amenable algebras. To define them, let us observe the following. If X is a left Banach A-module, then its dual

Banach space is a right Banach A-module with $(f \cdot a)(x) := f(a \cdot x)$. Similarly, the dual space to a right module is a left module. Thus we see that the dual space to a bimodule is itself a bimodule. Of course, there are much less dual bimodules, than arbitrary bimodules.

Definition 2. A Banach algebra A is called **amenable**, if its every derivation with values in an arbitrary **dual** A-bimodule is inner.

Why "amenable"? The very word means something like complaisant or obliging. But the main motivation is connected with the following, now classical theorem.

Johnson's Theorem. *Let G be a locally compact group. The standard group Banach algebra of G, that is, $(L_1(G), *)$, is amenable if and only if the group G is amenable as a locally compact group.*

After this theorem and the appearance of different approaches to the concept of the amenability it became clear that it is an exceptionally important class. For many concrete classes of Banach algebras that serve in various branches of analysis, the question of the description of amenable algebras within the class in question usually obtained a beautiful answer in terms of that area; I omit examples.

But one well known algebra stubbornly resisted. It was just our measure algebra $M(G)$. Earlier it was only known what happens in the case of an Abelian group. Using the Gel'fand representation of $M(G)$, Brown and Moran (1976) has shown that in the case of a continuous (= non-discrete) Abelian group $M(G)$ can not be amenable. But is it true if G is not Abelian, that is where is no Gel'fand representation? The question was open many years. It turned out that the problem was solved with the help of quite different considerations, based on homology.

When Johnson was studying his amenability, there was some activity in Moscow, outwardly without any relationship with what he was doing. There was an attempt to create a functional-analytic version of the homological theory of pure algebras. One of practical aims was to obtain new methods to compute the so-called Banach cohomology. We turn to the respective definition.

For Banach spaces $E_1, \dots, E_n, F$ we denote by $\mathcal{B}(E_1 \times \dots \times E_n, X)$ the Banach space of all bounded n-linear operators from $E_1 \times \dots \times E_n$ into X with the n-operator norm.

Let X A-mod-A. For every n $\mathbb{N}$ consider the space $\mathcal{B}(A \times \dots \times A, X)$ (with n copies of A), and denote it, for brevity, by $C^n(A, X)$. Then consider the operator $\delta^n : C^n(A, X) \to C^{n+1}(A, X) : f \mapsto \delta^n f$ (**"coboundary operator"**), where, for all a_k A, we have

$$\begin{aligned}\delta^n f(a_1, \dots, a_{n+1}) = a_1 \cdot f(a_2, \dots, a_{n+1}) \\ + \sum_{k=1}^{n} (-1)^k f(a_1, \dots, a_{k-1}, a_k a_{k+1}, \dots, a_{n+1}) \\ + (-1)^{n-1} f(a_1, \dots, a_n) \cdot a_{n+1}.\end{aligned}$$

Also we set $C^0(A,X) := X$ and $\delta^0 : C^0(A,X) \to C^1(A,X)$ with $[\delta^0(x)](a) := a \cdot x - x \cdot a$.

The sequence $0 \to C^0(A,X) \xrightarrow{\delta^0} C^1(A,X) \xrightarrow{\delta^1} \cdots$ is evidently a complex, called the **standard cohomology complex** for A and X.

Definition 3. The n-dimensional cohomology space of this complex, that is the quotient space $\mathrm{Ker}\,\delta^{n+1}/\mathrm{Im}\,\delta^n$, is called (according to the algebraic tradition) **n-dimensional cohomology group of A with coefficients in X.** We denote it by $\mathcal{H}^n(A,X)$.

We immediately see that 1-dimensional cohomology group is exactly $\mathcal{H}^1(A,X)$ in its previous meaning ("all derivations modulo inner derivations"). Other cohomology groups are also important; here I omit details. The computation of cohomology groups is an old typical problem, inherited from algebra.

For our needs the most important case is that of A-bimodules of the form $\mathcal{B}(X,Y)$ (cf. above). In this case the standard cohomology complex acquires some specific form. Namely, when we identify $C^n(A,\mathcal{B}(X,Y))$ with the space $\mathcal{B}(A \times \cdots \times A \times X, Y)$, briefly denoted by $\mathcal{B}^n(A,X,Y)$, we obtain the complex

$$0 \to \mathcal{B}(X,Y) \xrightarrow{\delta^0} \mathcal{B}^1(A,X,Y) \xrightarrow{\delta^1} \cdots, \qquad (\mathcal{B}(A,X,Y))$$

where new operators $\delta^n; n = 0,1,\ldots$ act as

$$\begin{aligned}\delta^n f(a_1,\ldots,a_{n+1},x) = a_1 \cdot f(a_2,\ldots,a_{n+1},x) \\ + \sum_{k=1}^{n}(-1)^k f(a_1,\ldots,a_{k-1},a_k a_{k+1},\ldots,a_{n+1},x) \\ + (-1)^{n-1} f(a_1,\ldots,a_n,a_{n+1}\cdot x)\end{aligned}$$

for $n > 0$, and $[\delta^0 f](a,x) := a \cdot f(x) - f(a \cdot x)$. Thus, *the space $\mathcal{H}^n(A,\mathcal{B}(X,Y))$ is the nth cohomology space of the complex $\mathcal{B}(A,X,Y)$.*

To compute the cohomology groups with the help of the standard complex is often a technical and tiresome task. But in algebra there are powerful methods that permit to avoid standard complexes. They are based on three main notions of homological algebra, that of **projective, injective and flat module.** So, if we wish that such methods would work in functional analysis, we must give right functional-analytic versions of these notions.

1. Preparing the stage

Now, for a time, we shall speak about left modules. But analogical definitions and constructions are valid for right modules and bimodules.

Let X, Y A-mod. The set of (bounded) morphisms between X and Y is denoted by $h(X,Y)$. It is a closed subspace of $\mathcal{B}(X,Y)$. Of course, in the complex $\mathcal{B}(A,X,Y)$ the space $\mathrm{Ker}\,(\delta_1)$ is not other thing that $h(X,Y)$.

Let φ $h(X,Y)$ and Z A-mod. Then we have two (bounded) operators $\varphi_\bullet : h(Z,X) \to h(Z,Y) : \psi \mapsto \varphi\psi$ and $\varphi^\bullet : h(Z,Y) \to h(Z,Y) : \psi \mapsto \psi\varphi$.

A complex $\cdots \leftarrow X_n \xleftarrow{d_n} X_{n+1} \leftarrow \cdots \quad (\mathcal{X})$ in A-mod is called **splitting**, if it has the so-called **contracting homotopy**; the latter is a sequence of morphisms $s_n : X_n \to X_{n+1}$ such that $s_n d_n + d_{n+1} s_{n+1} = \mathbf{1}$, where $\mathbf{1}$ denotes the relevant identity operator.

There is another, wider class of complexes, playing an outstanding role in Banach homology.

Definition 4. A complex of modules is called **admissible**, if it has a contracting homotopy in ***Ban*** (that is, consisting of operators that are not bound to be morphisms.)

We easily obtain

Proposition 1. *Every admissible, in particular, splitting complex is exact.*

Now take a complex $\mathcal{X}$ and one more module, say Y. Then two complexes in ***Ban*** appear, namely

$$\cdots h(Y, X_n) \xleftarrow{d_{n\bullet}} h(Y, X_{n+1}) \cdots \qquad (h(Y,\mathcal{X}))$$

and

$$\cdots h(X_n, Y) \xrightarrow{d_n^\bullet} (X_{n+1}, Y) \cdots . \qquad (h(\mathcal{X}, Y))$$

Proposition 2. *If $\mathcal{X}$ splits, then for every Y the complexes $h(Y,\mathcal{X})$ and $h(\mathcal{X},Y)$ are splitting (or, which is now the same, admissible) in **Ban** and hence exact.*

But if $\mathcal{X}$ is only admissible and does not split in A-mod, the previous assertion is no more true. Let

$$0 \longleftarrow X'' \xleftarrow{j} X \xleftarrow{i} X' \longleftarrow 0 \qquad (\mathcal{S})$$

be a short complex in A-mod. Then the short complex $h(Y,\mathcal{S})$ is always exact in the right and in the middle terms whereas the short complex $h(\mathcal{S},Y)$ is always exact in the left and in the middle term. However, the first complex is not bound to be exact in the left term, whereas the second one is not bound to be exact in the right term. In other words, the respective operators, denoted by $j_\bullet$ and $i^\bullet$, are not bound to be surjective.

We came to our principal triple definition.

Definition 5. (i) A left Banach A-module Y is called **projective**, respectively, **injective**, if for every short admissible complex $\mathcal{S}$ the complex $h(Y,\mathcal{S})$, respectively $h(\mathcal{S},Y)$, is exact. A right Banach A-module Y is called **flat**, if the left Banach A-module Y^* is injective.

Remark. If in this definition we would not consider only admissible short complex, we would obtain much less projective, injective and flat modules, and we could not create sufficiently rich homology theory.

Proposition 3. *If Y is projective, respectively injective, then for every "long" admissible complex $\mathcal{X}$ the complex $h(Y,\mathcal{X})$, respectively, $h(\mathcal{X},Y)$ is exact.*

It is important that *every projective module is flat.* As to examples and counterexamples, every closed ideal in $C(\Omega)$ with a metrizable compact space Ω is projective. One-dimensional modules over the same $C(\Omega)$ are flat, but not projective if Ω is connected. For the disk algebra $\mathcal{A}$ the complex plane $\boldsymbol{C}$, consider as an $\mathcal{A}$-module with $a \cdot \lambda := a(t)\lambda$, where $|t| < 1$, is not flat.

Remark. The difference between the rather rigid property of projectivity and much more widespread and flexible property of flatness is a very important phenomenon in functional analysis, inherited from algebra. In different areas of analysis it acquires quite different images. But I have no space to discuss it.

Now it is time to open my cards. The proof of our main theorem consists of three parts, roughly speaking, belonging to functional analysis, homology and harmonic analysis. Namely, we shall establish the following facts:

Theorem 1. *Let I be a closed right ideal in A, having a Banach complement subspace in A, or, equivalently, such that the short exact complex*

$$0 \longleftarrow A/I \xleftarrow{j} A \xleftarrow{i} I \longleftarrow 0 \tag{$\mathcal{I}$}$$

in A-mod is admissible. Suppose that the right A-module A/I is flat. Then I has a left bounded approximate identity.

Theorem 2. *For a right Banach A-module F the following properties are equivalent:*
(a) $\mathrm{Ext}^n(X,F^*) = 0$ *for all left Banach A-modules X and $n > 0$;*
(b) $\mathrm{Ext}^1(X,F^*) = 0$ *for all left Banach A-modules X;*
(c) *F is flat.*
As a corollary, all right Banach modules over an amenable Banach algebra are flat.

(Thus, to disprove the amenability of a given algebra it suffices to display at least one non-flat right module…)

Theorem 3. *The ideal $M_c(G)$ in $M(G)$ not only does not have the indicated approximate identity, but even does not coincide with its topological square $M_c(G)^2$ (that is, with the closure of the linear span of the set $\{\mu * \nu; \mu,\nu \quad M_c(G)\}$.)*

To prove this, we shall find in G special measurable subsets of rather exotic nature.

2. Outline of the proof of Theorem 1

Consider the dual complex $\mathcal{I}^*$ in A-mod. By our assumption, the short complex $h(\mathcal{I}^*,(A/I)^*$ is exact, and thus the operator $j^{*\bullet} : \varphi \mapsto \varphi j^*$ is surjective. Hence, taking the identity operator on $(A/I)^*)$, we see that j^* has a left inverse morphism, or, equivalently, i^* has a right inverse morphism, say $\rho : I^* \to A^*$. Consider the adjoint morphism $\rho^* : A^{**} \to I^{**}$. Since $A \subseteq A^{**}$, we can speak about the element $\widehat{e} := \rho^*(e)$. The classical theorem of Goldstein provides a bounded net in I, converging to $\widehat{e}$ in the weak * topology on I^{**}. Denote it by $e_\nu; \nu \quad \Lambda$.

It easy to show that for every $a \quad A$ the operator $S_a : I^{**} \to I^{**} : \alpha \mapsto \alpha \cdot a$ is weak*-continuous. Thus for every $a \quad I$ the net $e_\nu a = S_a(e_\nu)$ is weak*-convergent to $\widehat{e} \cdot a$, that is to $\rho^*(a)$. Therefore, since $I \subseteq I^{**}$ and $A \subseteq A^{**}$, we have $i^{**}(a) = i(a) = a$. Hence $\rho^*(a) = \mathbf{1}^*(a) = a$. This implies that the net $e_\nu a$ in I converges to a in the weak topology in I. It is known that the existence of such a net is equivalent to the existence of a left bound approximate identity.

Remark. This result, apart from being one of principal tools to prove our main theorem, has application to another area, the geometry of Banach spaces. In particular, it implies that the known Malliavin ideals in the Banach algebra $L_1(G)$, where G is Abelian, have no Banach complement.

3. Outline of the proof of Theorem 2

Let E be a Banach space. Consider the Banach space $A\widehat{\otimes}E$, where $\widehat{\otimes}$ denotes the **projective tensor product** of Banach spaces. It is a left Banach A-module with the outer multiplication well defined by $a \cdot (b \otimes x) = (ab) \cdot x$. It is called **free left Banach module with the base space** E**.** The universal property of $\widehat{\otimes}$ implies

Proposition 4. *For every $X \quad A$-mod there is a topological isomorphism*

$$I : h(A\widehat{\otimes}E, X) \to \mathcal{B}(E, X),$$

well defined by taking φ to $\psi : x \mapsto \varphi(e \otimes x)$.

Proposition 5. *Every free module is projective.*

If a bounded operator $\varphi : E \to F$ between Banach spaces is given, we can assign to it the operator, denoted by $\mathbf{1} \otimes \varphi : A\widehat{\otimes}E \to A\widehat{\otimes}F$ and well defined by $a \otimes x \mapsto a \otimes \varphi(x)$. Obviously, $\mathbf{1} \otimes \varphi$ is a morphism of left modules.

Proposition 6. *If $\mathcal{E}$ is a splitting complex of Banach spaces, then the complex $A\widehat{\otimes}\mathcal{E}$ (defined in an obvious way) is a splitting complex of Banach modules.*

For a given X and $n = 0, 1, \ldots$ we set $\boldsymbol{B}_n(X) := A\widehat{\otimes}(A\widehat{\otimes} \cdots \widehat{\otimes}A\widehat{\otimes}X)$, with n copies of A in brackets. We consider it as a free module with the base $A\widehat{\otimes} \cdots \widehat{\otimes}A\widehat{\otimes}X$. Further, for every n we consider the operator $d_n : \mathbb{B}_{n+1}(X) \to \mathbb{B}_n(X)$, well defined by

$$d_n(a \otimes a_1 \otimes \cdots \otimes a_{n+1} \otimes x) = aa_1 \otimes a_2 \otimes \cdots \otimes a_{n+1} \otimes x$$

$$+\sum_{k=1}^{n}(-1)^k a \otimes a_1 \otimes \cdots \otimes a_k a_{k+1} \otimes \cdots \otimes a_{n+1} \otimes x + (-1)^{n+1} a \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1} \cdot x.$$

Obviously, we obtain the complex $0 \longleftarrow \mathbb{B}_0(X) \xleftarrow{d_0} \mathbb{B}_1(X) \xleftarrow{d_1} \cdots \quad (\mathbb{B}(X))$.

Proposition 7. *The complex*

$$0 \longleftarrow X \xleftarrow{\pi} \mathbb{B}_0(X) \xleftarrow{d_0} \cdots, \qquad (0 \longleftarrow X \longleftarrow \mathbb{B}(X))$$

where $\pi : \mathbb{B}_0(X) \to X$ is well defined by $a \otimes x \mapsto a \cdot x$, is admissible.

We came to one of main notions of Banach homology, which has, as particular cases, the cohomology groups $\mathcal{H}(\cdot)$ and much more.

Let X, Y A-mod. Consider the complex $h(\mathbb{B}(X), Y)$.

Definition 6. The nth cohomology space of this complex is denoted by $\mathrm{Ext}^n(X, Y)$.

It turns out that we obtain our old acquaintances:

Theorem 4. *The space* $\mathrm{Ext}^n(X, Y)$ *coincides, up to a linear isomorphism, with* $\mathcal{H}^n(A, \mathcal{B}(X, Y))$.

Indeed, it is easy to show, using the universal property of the projective tensor product, that the complex $h(\mathbb{B}(X), Y)$ can be identified with $\mathcal{B}(A, X, Y)$.

Now observe that Theorem 2 is a direct corollary of the following theorem.

Theorem 5. *For a left Banach A-module Y the following properties are equivalent:*
(a) $\mathrm{Ext}^n(X, Y) = 0$ *for all left Banach A-modules X and $n > 0$*;
(b) $\mathrm{Ext}^1(X, Y) = 0$ *for all modules X*;
(c) Y *is injective.*

Proof. Thus, it remains to prove this theorem. Here the only not obvious implication is $(b) \Longleftarrow (c)$. Therefore we must show that for an arbitrary admissible complex $\mathcal{S}$ in A-mod (cf. Section 1), the operator $i^\bullet : h(X, Y) \to h(X', Y)$ is surjective. Why it is so?

For every n consider the sequence

$$0 \longleftarrow \mathbb{B}_n(X'') \xleftarrow{\mathbf{j}_n} \mathbb{B}_n(X) \xleftarrow{\mathbf{i}_n} \mathbb{B}_n(X') \longleftarrow 0 \qquad (\mathbb{B}_n(\mathcal{X}))$$

with $\mathbf{j}_n := \mathbf{1} \otimes j$, and $\mathbf{i}_n := \mathbf{1} \otimes i$. It is a splitting complex in A-mod. Therefore the complex $h(\mathbb{B}_n(\mathcal{X}), Y)$ is exact.

Now consider the commutative diagram

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & h(\mathbb{B}_0(X'), Y) \arrow[u] \arrow[r, "d_0^{\bullet\prime}"] & h(\mathbb{B}_1(X'), Y) \arrow[u] \arrow[r, "d_1^{\bullet\prime}"] & h(\mathbb{B}_2(X'), Y) \arrow[u] \arrow[r] & \ldots \\
0 \arrow[r] & h(\mathbb{B}_0(X), Y) \arrow[u, "i_0^\bullet"'] \arrow[r, "d_0^\bullet"] & h(\mathbb{B}_1(X), Y) \arrow[u, "i_1^\bullet"'] \arrow[r, "d_1^\bullet"] & h(\mathbb{B}_2(X), Y) \arrow[u, "i_2^\bullet"'] \arrow[r] & \ldots \\
0 \arrow[r] & h(\mathbb{B}_0(X''), Y) \arrow[u, "j_0^\bullet"'] \arrow[r, "d_0^{\bullet\prime\prime}"] & h(\mathbb{B}_1(X''), Y) \arrow[u, "j_1^\bullet"'] \arrow[r, "d_1^{\bullet\prime\prime}"] & h(\mathbb{B}_2(X''), Y) \arrow[u, "j_2^\bullet"'] \arrow[r] & \ldots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

We want to show that $\operatorname{Ker}(d_0^{\bullet\prime}) = i_0^{\bullet}(\operatorname{Ker}(d_0^{\bullet}))$. Our argument is an instructive (especially for non-algebraic audience) example of the so-called “diagram chase”, used when people work with big diagrams. Look: since our “vertical” complexes are exact, there exists $g_0 \quad h(\mathbb{B}_0(X), Y)$ with $i_0^{\bullet}(g_0) = f$. We have $i_1^{\bullet} d_0^{\bullet}(g_0) = d_0^{\bullet} i_0^{\bullet}(g_0) = d_0^{\bullet}(f) = 0$. Therefore $d_0^{\bullet}(g_0) = j_1^{\bullet}(h)$ for some $h \quad h(\mathbb{B}_1(X''), Y)$. Further, $j_2^{\bullet} d_1^*(h) = d_1^{\bullet} j_1^{\bullet}(h) = d_1^{\bullet} d_0^*(g_0) = 0$. Since $j_2^{\bullet}$ is injective, we have $d_1^{\bullet}(h) = 0$.

But $\operatorname{Ext}^1(X'', Y) = 0$, that is the lower horizontal complex is exact in term $h(\mathbb{B}_1(X''), Y)$. Therefore there is $l \quad h(B_0(X''), Y)$ with $d_0^*(l) = h$. Now set $g := g_0 - j_0^*(l)$. We easily see that $i_0^{\bullet}(g) = f$. At the same time

$$\begin{aligned} d_0^{\bullet}(g) &= d_0^{\bullet}(g_0) - d_0^{\bullet} j_0^{\bullet}(l) = d_0^{\bullet}(g_0) - j_1^{\bullet} d_0^{\bullet}(l) \\ &= d_0^{\bullet}(g_0) - j_1^{\bullet}(h) = d_0^{\bullet}(g_0) - d_0^{\bullet}(g_0) = 0, \end{aligned}$$

and we are done.

Identifying $h(A \widehat{\otimes} X', Y)$ with $\mathcal{B}(X', Y)$ and $h(A \widehat{\otimes} X, Y)$ with $\mathcal{B}(X, Y)$ (Proposition 4), we see that the operator $i_0^{\bullet}$ transforms to $\mathbf{i}^{\bullet} : \mathcal{B}(X, Y) \to \mathcal{B}(X', Y)$ which is an extension of $i^* : h(X, Y) = \operatorname{Ker}(d_0^{\bullet}) \to h(X', Y) = \operatorname{Ker}(d_0^{\bullet\prime})$. Thus the desired surjectivity of i^* is obtained. □

4. Outline of the proof of Theorem 3

Let G be a non-discrete locally compact group with the unit e. From now on we suppose that it is metrizable, since the general case can be reduced to the “metrizable” case.

Proposition 8. *There exists in G a decreasing sequence of subsets $K_1 \supseteq K_2 \supseteq K \supseteq \cdots$ with the following properties. Every K_n is a union of disjoint $^{n+1}$ sets $K_{ki_1\ldots i_n}; 1 \le k, i_1, \ldots, i_n \le$, referred as sets of level n. Each of summands is compact, has non-empty interior and diameter $< 1/(n+1)$. Further, we always have $K_{ki_1\ldots i_n, i_{n+1}} \subset K_{ki_1\ldots i_n}$. Finally, for every points $x_1, \ldots, x_4$, belonging to different sets of the same level, we have $x_1 x_2^{-1} x \ x_4^{-1} \neq e$.*

Theorem 6. *There exists a compact subset K in G (called* ***exotic****) with the following two properties:*

(i) *there is a positive measure $\Upsilon \quad M_c(G)$ with $Var(\Upsilon) = \Upsilon(K) = 1$ (“K is not so small”)*
(ii) *for every $s, t \quad G; s \neq t$ the set $sK \cap tK$ has no more than 3 points (“K is not large”).*

The desired set is $K := \cap_{n=1}^{\infty} K_n$. The proof of (i) uses that $M(G)$ is a dual Banach space, and hence we can apply Banach–Alaoglu Theorem.

Proposition 9. *Let $K \subset G$ be exotic. Then for all $\mu, \nu \quad M_c(G)$, $\mu * \nu(K) = 0$.*

Showing this, we use the following claim: the number $|\nu|(sK) > 0$, where $|\nu|$ is the total variation of μ, is not zero only for at most countable set of s in G.

Proposition 7 implies that the nonzero functional $f : M_c(G) \to \boldsymbol{C} : \mu \mapsto \mu(K)$ vanishes on $M_c(G)^2$. Theorem 3 immediately follows.

Combining this with Theorem 1, that is with already accomplished Part 2 of our plan, we see that the quotient right $M(G)$-module $M(G)/M_c(G)$ is not flat. So, to complete the proof of our main theorem, we must do two things: (1) to show that amenability implies flatness of modules, that is, do Part 1 of our plan, and (2) to show that sets K with described properties indeed exist, that is to complete Part 3. At first let us return to Banach homology.

Armed with this proposition, we proceed to construct a special compact set in G. As the first step, take in that proposition $N :=\ $, $\varepsilon = 1/2$ and consider the respective sets $U_1, \dots, U_4$. We obtain $\ ^2$ open sets U_{ki} with indicated properties and diameters $< 1/2$. Set $K_{ki} := \overline{U_{ki}}$ and call them *sets of level 1.* Also set $K_1 := \cup_{ki} K_{ki}$. Then, as the second step, take $N :=\ ^2$ and U_{ki} as given sets. Then the previous proposition provides us with open sets $U_{ki_1i_2}; k, i_1, i_2 \quad \{1, \dots, \ \}$ with indicated properties and diameters $< 1/3$. Set $K_{ki_1i_2} := \overline{U_{ki_1i_2}}$ and call them *sets of level 2.* Also set $K_2 := \cup_{ki_1i_2} K_{ki_1i_2}$. Then, as the third step, take $N :=$ and $U_{ki_1i_2}$ as given sets, and so on.

Thus we obtain the decreasing sequence of sets $K_1 \supseteq K_2 \supseteq K \ \supseteq \dots$ Every K_n is a union of disjoint $\ ^{n+1}$ sets $K_{ki_1\dots i_n}$ "of level n". Each of summands is compact, has non-empty interior and diameter $< 1/(n+1)$. Set $K := \cap_{n=1}^{\infty} K_n$. Being the intersection of a decreasing sequence of non-empty compact sets, K is itself compact and non-empty. (Prove, as an exercise, that its cardinality is at least continuum).

Proposition 10. *For every 4 distinct points $x_1, \dots, x_4 \quad K$, we have*

$$x_1 x_2^{-1} x \ \ x_4^{-1} \neq e.$$

Proof. Take $x_k : k \quad \{1, \dots, \ \}$ and consider the decreasing sequence of sets of level $n = 1, 2, \dots$, containing x_k. Denote these sets, for brevity, by $K(k, n)$. Since diameters of sets $K(k, n); n = 1, 2, \dots$ converges to 0, x_k is the only point, belonging to all $K(k, n)$. It follows that for every distinct $k, l \quad \{1, \dots \ \}$ we have $K(k, n_{k,l}) \neq K(l, n_{k,l})$ for some natural $n_{k,l}$: otherwise x_k and x_l would coincide. Taking $n := \max\{n_{k,l}\}$, we see that our x_k belong to different sets of level n. It remains to apply Proposition 8. $\square$

Now, taking into account what was said in §3, see that in order to complete the proof of the main theorem, we must show that the constructed set K is indeed of middle size. To begin with, we must construct the measure Υ, mentioned in the respective definition.

Denote by $\mathbf{m}$ the Haar measure on G, to be definite, left-invariant. Choose an arbitrary set L of level $n \quad \mathbb{N}$. Since it has non-empty interior, we have $\mathbf{m}(L) > 0$. Introduce the measure Υ_L on G, setting, for a Borel set $M \subseteq G$,

$$\Upsilon_L(M) := \mathbf{m}(M \cap L) / \ \ ^{n+1} \mathbf{m}(L).$$

Further, introduce the measure Υ_n on G as a sum of measures Υ_L, taken for all (pairwise disjoint, as we remember) sets L of level n. Obviously, we have $\Upsilon_n(K_n) = 1$ and $\Upsilon_n(G \setminus K_n) = 0$.

We see that the measures $\Upsilon_n; n \quad \mathbb{N}$ belong to the unit sphere of $M(G)$. But the latter is a dual space, namely, $M(G) = C_0(G)^*$ and hence it can be considered with the weak* topology. By the Banach–Alaoglu Theorem, the sequence Υ_n has in the latter topology an accumulation point. Denote it by Υ; of course, it is a positive measure, and $Var\Upsilon$, that is, $\Upsilon(G)$ or $\|\Upsilon\|$, is ≤ 1.

The following proposition provides the property (i) of sets of "middle size", that is, exotic.

Proposition 11. (i) *The measure* Υ *is continuous.*
(ii) $\Upsilon(K) = 1$*, and* $\Upsilon(G \setminus K) = 0$*.*

Proof. (i) We must show that $\Upsilon(\{x\}) = 0$ for all $x \quad G$. Suppose that $x \quad K$; the easier case, when $x \quad G \setminus K$, I leave to the listeners. Then for every n our x belongs to some set, say $K(x,n)$ of level n. Fix some n and consider $f_n \quad C_0(G)$ with range $[0,1]$ such that $f_n(x) = 1$ and $f_n = 0$ outside $K(x,n)$. We have, of course, $\Upsilon(\{x\} \leq \int_G f_n(t)d\Upsilon(t)$. By the choice of Υ, $\int_G f_n(t)d\Upsilon(t)$ is an accumulation point of numbers $\int_G f_n(t)d\Upsilon_m(t); m \quad \mathbb{N}$. For each $m \geq n$ we obviously have that the latter integral is the sum of numbers $\int_L f_n(t)d\Upsilon_m(t)$, taken for all sets L of level m, contained in $K(x,n)$. But there are $\ {}^m/\ {}^n$ of such summands, and each of them is $\leq 1/\ {}^m$. Hence for all $m \quad \mathbb{N}$ we have $\int_G f_n(t)d\Upsilon_m(t) \leq 1/n^4$, and the same is true to the respectively accumulation point of these numbers, that is for $\int_G f_n(t)d\Upsilon(t)$. Therefore $\Upsilon(\{x\} \leq 1/\ {}^n$ for all n, and thus $\Upsilon(\{x\}) = 0$.

Now suppose that $x \ / \ K$. Then $x \ / \ K_m$ for all $m \geq$ some n. Take $f_n \quad C_0(G)$ with the range $[0,1]$ and such that $f_n(x) = 1$ and $f_n = 0$ on K_n and hence for all $K_m; m \geq n$. Then for all $m \geq n$ we have

$$\int_G f_n(t)d\Upsilon_m(t) \leq \int_{K_m} f_n(t)d\Upsilon_m(t) = 0,$$

hence the same is true, when we replace Υ_m by Υ. Thus

$$\Upsilon(\{x\}) \leq \int_G f_n(t)d\Upsilon(t) = 0.$$

(ii) Since $Var\Upsilon(G) \leq 1$, we have $\Upsilon(K) \leq 1$ and $\Upsilon(G \setminus K) \leq (1 - \Upsilon(K)$. Therefore it is sufficient to show that $\Upsilon(K) \geq 1$. Since $K = \cap_{n=1}^{\infty} K_n$, we have $\Upsilon(K) = \lim_{n\to\infty} \Upsilon(K_n)$, and therefore it suffices to show that $\Upsilon(K_n) \geq 1$ for all n.

Fix, for a moment, n. We remember that for all $f \quad C_0(G)$ the integral $\int_G f(t)d\Upsilon(t)$ is accumulation point of numbers $\int_G f(t)d\Upsilon_m(t); m \quad \mathbb{N}$. Take an arbitrary non-negative $f \quad C_0(G)$ with $f = 1$ on K_n and hence $f = 1$ on all $K_m; m \geq n$. For all these m we have $\int_G f(t)d\Upsilon_m(t) \geq \Upsilon_m(K_m) = 1$. Consequently, $\int_G f(t)d\Upsilon(t)$, being an accumulation point of these numbers, is also ≥ 1. It remains to remind that $\Upsilon(K_n)$ is the infimum of all numbers $\int_G f(t)$ $C_0(G)$ with $f \geq 0$ and $f = 1$ on K_n. □

Thus all what we still have to do is to show that K satisfied the property (ii) of sets of middle size.

Proposition 12. *For every s, t $G; s \neq t$, the set $sK \cap tK$ has no more than three points.*

Proof. Of course, it suffices to show that $sK \cap K; s \neq e$ has no more than three points. Let x $sK \cap K$; then $x = sy$ for some y K. We shall show that every point in $sK \cap K$ must coincide with one of points x, y, sx. Suppose that, on the contrary, there is z $K \cap sK$, different from indicated three points. We have $z = su$ for some u K. Observe that $yx^{-1}zu^{-1} = yy^{-1}s^{-1}suu^{-1} = e$. Hence, by Proposition 10, some of points x, y, z, u coincide. But we have $x, y \neq z$, $x \neq y$ and $z \neq u$ since $s \neq e$, $x \neq u$ because otherwise $sx = su = z$, and finally $y \neq u$ because otherwise $x = sy = su = z$. We came to a contradiction. □

The proof of our main theorem is completed.

Summing up

Theorems 3 and 1 combined show that the right $M(G)$ module $M(G)/M_c(G)$ is not flat. By Theorem 2, this implies that $M(G)$ is not amenable, that is our main theorem is true.

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Functional Analysis techniques in Optimization and Metrization problems

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We give an introduction on the main subjects of metric geometry, derived (conceptually) from the Riemannian theory, in the setting of Lie groups groups G modeled by locally convex spaces, and admitting continuous Finsler metrics. The focus is put in the functional analysis techniques, since such norms are usually not differentiable and the variational calculus is not at hand. These techniques, however, allow us to retain some of the fine results of the tensor calculus, and even in the absence of linear connections, we show in the setting of Lie groups with a bi-invariant metric, how some results such as the minimality of one-parameter groups, can be recovered using such techniques.

1. Introduction

We begin this abstract by recalling some general definitions in the setting of smooth manifolds: manifolds in these notes are modeled with charts in a Hausdorff locally convex topological vector space (shortly l.c.s.). The differential of a map $f : M \to N$ among smooth manifolds will be denoted by $f_* : TM \to TN$ and its specialization by $f_{*p} : T_pM \to T_{f(p)}N$, $p \quad M$.

A *Lie group* G is a manifold such that the operation $(x, y) \mapsto xy^{-1}$ is smooth (at least C^2) as a map $G \times G \to G$. If $g \quad G$ and $L_g : h \mapsto gh$ denotes the left multiplication in G, we denote $gv = (L_g)_{*h}v \quad T_{gh}G$ for $h \quad G$, $v \quad T_hG$. We denote $1 \quad G$ the identity of the group and $\mathrm{Lie}(G) = T_1G$ its Lie algebra. The Lie bracket in $\mathrm{Lie}(G)$ will be denoted by $[\cdot, \cdot]$: it is always a bi-linear, anti-symmetric and continuous map. If $c_g(h) = ghg^{-1}$ is the conjugation automorphism, i.e. $c_g = L_gR_g^{-1}$ for $g \quad G$, we follow the standard notation $\mathrm{Ad}_g = (c_g)_{*1}$

with $\mathrm{Ad} : G \to \mathrm{GL}\big(\mathrm{Lie}(G)\big)$ a group homomorphism. If $\mathrm{Lie}(G)$ is not a Banach space then $\mathrm{GL}\big(\mathrm{Lie}(G)\big)$ is not necessarily a Lie group, but it is a subgroup of the space of diffeomorphisms of $\mathrm{Lie}(G)$ therefore there is a natural notion of smoothness. We denote $\mathrm{ad} = (\mathrm{Ad})_{*1} : \mathrm{Lie}(G) \to \mathcal{L}\big(\mathrm{Lie}(G)\big)$ which is a linear Lie algebra morphism, and in fact $\mathrm{ad}(v)(w) = [v, w]$ for any $v, w \quad \mathrm{Lie}(G)$ (see Neeb [11, Section II.3]).

Definition 1 (Finsler norms and semi-norms). Let E be a l.c.s., $\mu = |\cdot| : E \to \mathbb{R}_{\geq 0}$ a continuous function. Then μ is a *Finsler norm* if it is sub-additive and positively homogeneous: $|v + w| \leq |v| + |w|$ and $|\lambda v| = \lambda|v|$ for $v, w \quad E$ and $\lambda \quad \mathbb{R}_{\geq 0}$, and $|v| = 0$ implies $v = 0$. If $|tv| = |t|\,|v|$ for all $t \quad \mathbb{R}$, we obtain the standard notion of continuous vector space norm.

Definition 2 (Finsler metrics for TM). Let M be a manifold modeled by a l.c.s E. Let $\mu : TM \to \mathbb{R}_{\geq 0}$ be a selection of a tangent Finsler semi-norm $\mu_p = |\cdot|_p : T_pM \to \mathbb{R}_{\geq 0}$, for each $p \quad M$, such that $\mu : TM \to \mathbb{R}$ is a continuous map.

Definition 3 (Rectifiable paths and length). We say that a curve $\alpha : [a, b] \to M$ is *rectifiable* if α is differentiable a.e. in some chart of M and $t \mapsto |\dot\alpha(t)|_{\alpha(t)}$ is Lebesgue integrable. For piecewise smooth or rectifiable arcs $\alpha : [a, b] \to M$, define the *length* of α as

$$\mathrm{Length}_\mu(\alpha) = \int_a^b |\dot\alpha(t)|_{\alpha(t)}\, dt.$$

Definition 4. For $x, y \quad M$, consider the infimum of the lengths of such arcs joining x, y in M,

$$\mathrm{dist}_\mu(g, h) = \inf\big\{\,\mathrm{Length}_\mu(\alpha) : \alpha : [0, 1] \to M \text{ rectifiable}, \alpha(0) = x, \alpha(1) = y\big\}.$$

Then $\mathrm{dist}_\mu : M \times M \to \mathbb{R}_{\geq 0}$ is a p.s.d. (pseudo-quasi-distance): it is finite in each arc-wise connected component of M, it obeys the triangle inequality, and it is reversible. This gives place of what is called an inner length space, a metric space where the distance can be recovered by means of the infimum of the paths joining the given endpoints. A good reference on the subject is the book by Burago, Burago and Ivanov [2].

Remark 5. The matter of whether $\mathrm{dist}_\mu(x, y) = 0$ implies $x = y$ in M is more subtle. There are examples when this fails, see Michor and Mumford [9] for such an example; see also the paper by Clarke [3].

Definition 6. We will denote with (M, dist_μ) the underlying (pseudo-quasi) metric space. Nevertheless, this distance or quasi-distance induces a topology in M, and we will refer to the topology induced as τ_μ when needed; otherwise the topology of M will always be the manifold topology denoted by τ_M. Clearly, τ_μ will be Hausdorff if and only if dist_μ is non-degenerate. It is apparent that $\mathrm{dist}_\mu : (M, \tau_M) \times (M, \tau_M) \to \mathbb{R}$ is continuous, and τ_μ is finer than τ_M.

2. Metrics, connections, exponential map

For a given variation $\nu : (-\varepsilon, \varepsilon) \times [a, b] \to M$ of a fixed path $\gamma : [a, b] \to M$, we say that the variation has fixed endpoints if $\nu(s, a) = x$ and $\nu(s, b) = y$ for all $|s| < \varepsilon$; for such variations the path γ is an *extremal* of the length functional if the function

$$f(s) = \mathrm{Length}_\mu(\nu_s) = \int_a^b \left|\dot\nu(s,t)\right|_{\nu(s,t)} dt$$

has zero derivative (or one sided derivative) for $s = 0$. Here and in what follows we denote by $\frac{d}{dt} = (\quad)^{\cdot}$ the time derivative.

It is well-known that when the metric $|\cdot| = g$ is Riemannian, extremal paths obey Euler differential equation, $\nabla_{\dot\gamma}\dot\gamma = 0$. Here ∇ is the Levi-Civita connection of the Riemannian metric, which is the unique operator defined in vector fields of $\mathfrak{X}(M)$, that is $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ such that ∇ is $\mathbb{R}$-bilinear,

1. $\nabla_{fX}Y = f\nabla_X Y$ for all $f \quad C^\infty(M)$,
2. $\nabla_X(fY) = X(f)Y + f\nabla_X Y$, where $X(f)(p) = f_{*p}(X_p)$,
3. $\nabla_X Y - \nabla_Y X = [X, Y]$,
4. $Z(\langle X, Y\rangle) = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle$.

Usually, in a chart (φ, U) of M, the connection is given by a bilinear operator Γ such that $\nabla_X Y = Y_* X - \Gamma(X, Y)$. In the finite dimensional setting, for a given orthonormal basis $(e_i)_i$ of TM (defined locally), the coefficients of this bilinear operator $\Gamma(e_i, e_j)_k = \Gamma^k_{ij}$ are just the Christoffel symbols of second kind of the given Riemannian metric. The details on these standard considerations can be found, for instance, in Lang's book [7].

Solutions of Euler's equations are called geodesics of the Riemannian manifold (M, g); in Banach manifolds, and by the theorem of existence and uniqueness of ordinary differential equations, for each pair $(p, v) \quad TM$ there exists a uniquely determined geodesic $\gamma_{(p,v)} : (-\varepsilon, \varepsilon) \to M$ defined in a neighborhood of $t = 0$, such that $\gamma(0) = p$ and $\dot\gamma(0) = v$. By virtue of Euler's equation and the properties of the connections, geodesics have nice reparameterization properties, $\gamma_{(p,sv)}(t) = \gamma_{(p,v)}(t)$ for all s, t where both sides make sense. This enables the definition of Riemannian exponential $\exp : TM \to M$ (in fact, only defined in a neighborhood of the zero section of the fiber tangent bundle TM), such that

$$\exp(p, tv) = \gamma_{(p,v)}(t) := \exp_p(tv),$$

where $\exp_p : T_pM \to M$ denotes the restriction of exp to T_pM (again with the precaution that in principle it is only defined in a neighborhood of $0 \quad T_pM$). By virtue of the smoothness of the solutions with respect to the initial conditions, and the reparameterization property, it is clear that $(\exp_p)_{*0} = \mathrm{id}_{T_pM}$, hence in the setting of Banach manifolds $\exp_p$ gives a diffeomorphism of a neighborhood B of $0 \quad T_pM$ (usually taken as some ball of radius r), and an open neighborhood V of $p \quad M$. Its inverse φ_p is called an *exponential chart* (φ_p, V) of M around p.

§ We remark that general smooth manifolds (modeled by locally convex spaces) may admit Riemannian metrics (usually called weak, because they are only continuous bilinear maps in each tangent space, and do not give the original topology of T_pM), but whether there exists a Levi-Civita connection for such metric, or an exponential map for that metric, depends on each particular case and must be obtained (if they exist) with ad-hoc methods. When there exists a Riemannian exponential, but outside the setting of Banach manifolds, the fact that $(\exp_p)_{*0} = \mathrm{id}_{T_pM}$ does not indicate the existence of an exponential chart, since the inverse mapping theorem does not apply.

2.1. Lie groups

As usual left invariant vector fields on a Lie group induce the Lie bracket of the Lie algebra, by means of $[v,w] = [X_v, X_w](1)$, where $1 \quad G$ is the identity of G and the bracket on the right-side is the usual Lie bracket of vector fields on manifolds. As mentioned in the introduction, the Lie bracket can also be computed differentiating the adjoint representation Ad at the identity of G.

Again, outside the realm of Banach–Lie groups, solution of the flow equation for left invariant vector fields $X_v(g) = gv \ (= (L_g)_{*1}v)$ does not necessarily exist; however by means of standard logarithmic derivative techniques, if it exists, it is unique (and gives place to the one-parameter group $\gamma_v : \mathbb{R} \to G$, usually written as $\gamma_v(t) = e^{tv}$). The exponential map of the group is defined in this case by means of $\exp(v) = e^v = \gamma_v(1)$, and clearly $\exp : \mathrm{Lie}(G) \to G$. By means of the flow equation $X_v(\gamma(t)) = \dot\gamma(t)$ with initial condition $\gamma(0) = 1$, it follows that if exp is defined for all $v \quad \mathrm{Lie}(G)$, and it is smooth, then $(\exp)_{*0} = \mathrm{id}_{T_1G}$. Again, for Banach-Lie groups this shows that (inverse of) the exponential can be used as a chart for G, but this is no longer true for a locally convex group; such groups with smooth exponential which is a local diffeomorphism are called *exponential*.

Remark 7. If G carries a smooth exponential function, then its derivative can be computed explicitly: for $v, w \quad \mathrm{Lie}(G)$,

$$\exp_{*w}(v) = e^w \int_0^1 \mathrm{Ad}_{e^{-sw}}\, v\, ds = e^w \int_0^1 e^{-s\ \mathrm{ad}\, w} v\, ds.$$

Remark 8 (Left-invariant and bi-invariant metrics). Let G be a Lie group, we fix $|\cdot|$ a Finsler norm in $\mathrm{Lie}(G)$ and define $|v|_g = |(L_g)_{*1}^{-1} v|$ for $v \quad T_gG$, then the group G has a *left-invariant Finsler metric* $|\cdot|_g : T_gG \to \mathbb{R}_{\geq 0}$, because if $g, h \quad G$ then

$$|hv|_{gh} = \left|(gh)^{-1}hv\right| = \left|g^{-1}v\right| = |v|_g \quad \text{ for } v \quad T_gG,$$

and the map $(g,v) \mapsto |v|_g = |g^{-1}v|$ is continuous as a map from TG to $\mathbb{R}$. Any left-invariant Finsler metric in G can be obtained with this procedure. Note also that $\left|\mathrm{Ad}_g v\right|_1 = \left|vg^{-1}\right|_{g^{-1}} = |v|_1$ when the metric is also right-invariant. In that case we say that the metric is *bi-invariant*.

3. Unitary groups of Hilbert space operators

If a group G admits a bi-invariant Riemannian metric g, it is well-known that geodesics δ of (G, g) are left-translations of one-parameter groups: $\delta(t) = ue^{tv}$ (cf. [4, 2.90]).

If the metric is not Riemannian, it is also expected that one-parameter groups will be short paths for the rectifiable distance. This was established using spectral theory and certain functional analysis techniques in [1] for special groups of unitary operators acting on a Hilbert space H, i.e. for a given Banach ideal $\mathcal{I} \subset \mathcal{K}(H)$ of compact operators, with norm $|\cdot|_{\mathcal{I}}$, we consider the skew-adjoint part of $\mathcal{I}$ denoted $\mathcal{I}_{sa}$, and the special unitary group

$$U_{\mathcal{I}}(H) = \{u^* = u^{-1} \quad U(H) : u - 1 \quad \mathcal{I}\} = \exp(\mathcal{I}_{sa}),$$

following P. de la Harpe [6]. The main theorem there is that the one-parameter group $\delta(t) = e^{tz}$, with $z \quad \mathcal{I}_{sa}$ and $\|z\|_\infty \leq \pi$, is shorter than any other piecewise smooth map joining $1, u = e^z$ in $U_{\mathcal{I}}(H)$, therefore $\mathrm{dist}_{\mathcal{I}}(1, u) = |z|_{\mathcal{I}}$. To obtain those results, the formula in Remark 7 is of great relevance, but the key functional analytic tool is the knowledge of certain properties of the spectrum of the product of two unitary operators (in relation with the spectrum of both of them separately). These ideas are connected to the conjecture of Horn on the spectra of the sum of two hermitian operators, and what is called the honeycomb conjecture (see [5] and the references therein).

As mentioned, this result extends what is well-known for bi-invariant Riemannian metrics on Lie groups. But it also extends a result on the group U of unitary operators of a C^*-algebra, equipped with the Finsler norm obtained by putting the uniform norm (the spectral norm of the algebra) in each tangent space. This was obtained by Porta and Recht in [13], using a technique that involves representing paths in U as paths in the unit sphere S of a Hilbert space H (usually, the space where the C^*-algebra is represented as a subset of $B(H)$. In this construction, one-parameter groups in U are mapped isometrically onto great arcs in the unit sphere S, which is a Riemannian manifold with well-known geodesics: its great arcs. Therefore the minimality of geodesics in the (highly non-Riemannian) Finsler manifold U is obtained by means of knowing the Riemannian geometry of the Hilbert space sphere, and some standard results on representation theory of bounded operators.

To end this paper, we remark that similar results *for any exponential Lie group* with a continuous bi-invariant Finsler metric, were recently obtained in [8]. In this case, the techniques used for the proof are closely related to the theory of dissipative operators in a Banach space.

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Twistor Geometry and Gauge Fields

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In our course we have presented the basics of twistor theory and its applications to the solution of Yang–Mills duality equations. The first part describes the twistor correspondence between geometric objects in Minkowski space and their counterparts in twistor space. In the second part we apply twistor theory to the study of Yang–Mills duality equations on $\mathbb{R}^4$. We include a list of references for further study.

1. Twistor model of Minkowski space

We start with the geometry of Minkowski space M provided with the action of the Lorentz group. The main geometric objects are the light lines (light rays) and light cones together with their complex analogues. Complexified Minkowski space $\mathbb{C}M$ contains both M and its Euclidean counterpart E. We also make use of the future tube $\mathbb{C}M_+ = M + iV_+$ (V_+ is the future light cone) which is an open subset in $\mathbb{C}M$.

The Pauli map associating with a vector $x = (x^0, x^1, x^2, x\ )\quad M$ the Hermitian matrix

$$X := \sum_{\mu=0} x^\mu \sigma_\mu,$$

where $\sigma_0 = I$, σ_i, $i = 1, 2, 3$, are Pauli matrices, realizes M as the space Herm(2) of Hermitian 2×2-matrices and $\mathbb{C}M$ as the space $\mathbb{C}[2 \times 2]$ of complex 2×2-matrices. Under this map the Lorentz norm of $x \quad M$ is sent to $\det X$. The group $\mathrm{SL}(2, \mathbb{C})$ acts naturally on Herm(2) and is a double cover of the Lorentz group.

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The future tube $\mathbb{C}M_+$ under Pauli map is transformed into the matrix upper halfplane

$$H_+ = \left\{ Z \quad \mathbb{C}[2 \times 2] : \ \mathrm{Im}\ Z := \frac{1}{2i}(Z - Z^*) > 0 \right\}$$

where the inequality $\mathrm{Im}\ Z > 0$ means that the Hermitian matrix $\mathrm{Im}\ Z$ is positive definite. The space $\mathbb{C}^2$, provided with the action of the group $\mathrm{SL}(2, \mathbb{C})$, is called the spinor space.

The twistor space $\mathbb{T}$ is the 4-dimensional complex vector space with coordinates written in the form $\zeta = (\omega, \pi)$ with $\omega, \pi \quad \mathbb{C}^2$. Associate with a matrix $Z \quad \mathbb{C}[2 \times 2]$ the 2-dimensional complex subspace in $\mathbb{T}$ determined by the system of two complex homogeneous equations: $\omega = Z\pi$. This map defines an embedding of the space $\mathbb{C}[2 \times 2]$ into the Grassmann manifold $G_2(\mathbb{T})$ of 2-dimensional complex subspaces in $\mathbb{T}$. Taking its composition with the Pauli map we obtain the embedding

$$\mathbb{C}M \longrightarrow \mathbb{C}[2 \times 2] \longrightarrow G_2(\mathbb{T})$$

of the complexified Minkowski space $\mathbb{C}M$ into the Grassmannian $G_2(\mathbb{T})$. Since $G_2(\mathbb{T})$ is compact it is natural to consider it as a model of compactified complexified Minkowski space $\mathbb{CM}$. The projectivization $\mathbb{PT}$ of the twistor space $\mathbb{T}$ is called the space of projective twistors. We can also consider the Grassmannian manifold $G_2(\mathbb{T})$ as the space $G_1(\mathbb{PT})$ of projective lines in $\mathbb{PT}$. The composite map $\mathbb{C}M \to G_2(\mathbb{T}) = G_1(\mathbb{PT})$ is called the twistor transform or Penrose correspondence.

2. Twistor correspondence

Consider first the properties of twistor correspondence in the case of complex Minkowski space. By twistor transform a point of $\mathbb{C}M$ is sent to a projective line in $\mathbb{PT}$. On the other hand, a point in $\mathbb{PT}$ corresponds to a light plane in $\mathbb{C}M$ called α-plane (light plane is the plane generated by the pair of light lines). In dual way, a projective plane in $\mathbb{PT}$ corresponds to a light plane in $\mathbb{C}M$ called β-plane. It implies that a complex light line (which is the complexification of light line) is sent to a $(0, 2)$-flag in $\mathbb{PT}$ consisting of a point in $\mathbb{PT}$ and projective plane containing this point.

Switch now to the case of real Minkowski space M. Denote by $\Phi(\zeta)$ the norm of a twistor $\zeta = (\omega, \pi) \quad \mathbb{T}$ given by $\Phi(\zeta) = \mathrm{Im} < \omega, \pi >$ where $< \omega, \pi >$ is the Hermitian product of vectors $\omega, \pi \quad \mathbb{C}^2$. Denote by $\mathbb{N}$ the quadric in $\mathbb{T}$ given by the equation $\mathbb{N} = \{\zeta \quad \mathbb{T} : \ \Phi(\zeta) = 0\}$ and by $\mathbb{PN}$ the associated projective quadric. The points of M under twistor transform are sent to the projective lines lying in $\mathbb{PN}$. On the other hand, a light line in M corresponds to a point of $\mathbb{PN}$. So in the case of M we have the following duality: points of M correspond to projective lines in $\mathbb{PN}$ and light lines in M correspond to points of $\mathbb{PN}$. We see that the light lines, which can intersect in M, split into separate points of $\mathbb{PN}$. This fact is of fundamental importance for the twistor theory.

The quadric $\mathbb{N}$ divides the twistor space $\mathbb{T}$ into two parts. Denote them by $\mathbb{T}_\pm = \{\zeta \quad \mathbb{T} : \ (\pm 1)\Phi(\zeta) > 0\}$ and by $\mathbb{PT}_\pm$ the corresponding projective subsets. A point of the future tube $\mathbb{C}M_+$ under twistor transform is sent to a projective line contained in $\mathbb{PT}_+$. The quadric $\mathbb{N}$ has the signature (2,2) and the group $\mathrm{SU}(2,2)$ of linear transformations of $\mathbb{T}$, preserving this quadric, is a 4:1 covering of the group of conformal transformations of M.

We turn now to the case of Euclidean space E. A point of E under twistor transform is sent to the projective line in $\mathbb{PT}$ which is invariant under the map $j : [\zeta_1 : \zeta_2 : \zeta \ : \zeta_4] \longmapsto [-\zeta_2 : \zeta_1 : -\zeta_4 : \zeta \ ]$. In the Euclidean case the twistor transform coincides with the Hopf bundle

$$\pi : \mathbb{CP} \ \xrightarrow{\mathbb{CP}^1} \mathbb{E}$$

where $\mathbb{E}$ is the *compactified Euclidean space* equal to the sphere S^4 and the fibers of π are precisely the j-invariant projective lines.

The main idea of Penrose twistor program is that under twistor transform solutions of conformally invariant equations of field theory in $\mathbb{M}$ should correspond to the objects of complex algebraic geometry in $\mathbb{PN}$.

3. Instantons and Yang–Mills fields

Let X be a compact oriented Riemannian 4-manifold and G is the gauge group being a compact Lie group (e.g. $G = \mathrm{SU}(2)$) with Lie algebra $\mathfrak{g}$. Let $P \to X$ is a principal G-bundle on X and A is a gauge potential on X given a 1-form $A \quad \Omega^1(X, \operatorname{ad} P)$ with values in the adjoint bundle $\operatorname{ad} P = P \times_G \mathfrak{g}$. Denote by D the exterior covariant derivative associated with A. Then $F = DA$ is the gauge field generated by A.

The Yang–Mills action is given by the formula

$$S(A) = \frac{1}{2}\int_X \|F\|^2 \mathrm{vol}$$

where the norm $\|\cdot\|^2$ is the inner product on differential forms with values in $\mathfrak{g}$, generated by the Riemannian metric on X and an invariant inner product on $\mathfrak{g}$, vol is the volume element on X. The Yang–Mills field is a critical point of the functional S being a solution of the Euler–Lagrange equations. They have the form $D^*F = 0$ (D^* is the adjoint operator of D) and are called the Yang–Mills equations. They can be also written in the form $D(\star F) = 0$, where $\star$ is the Hodge $\star$-operator.

A gauge field F is called selfdual (resp. anti-selfdual) if $*F = F$ (resp. $*F = -F$). Due to Bianchi identity $DF = 0$, solutions of the duality equations $*F = \pm F$ satisfy automatically the Yang–Mills equations. By writing F in the form $F = F_+ + F_-$ where $F_\pm = \frac{1}{2}(*F \pm F)$ we can rewrite Yang–Mills functional as

$$S(A) = \frac{1}{2}\int_X \|F_+\|^2 + \|F_-\|^2)\mathrm{vol}.$$

The topological charge of F is given by the formula

$$S(A) = \frac{1}{8\pi^2} \int_X \|F_+\|^2 - \|F_-\|^2)\mathrm{vol}.$$

Comparing the last two formulas we see that

$$S(A) \geq \quad \pi^2|k|$$

and the equality here is attained precisely on solutions of the duality equations. In other words, solutions of the duality equations yield local minima of S. Instantons (resp. anti-instantons) are anti-selfdual (ASD)(resp. selfdual) solutions of duality equations with finite Yang–Mills action. The moduli space of instantons is the quotient of the space of instantons modulo gauge transformations.

4. Atiyah–Ward theorem

We specify now to the case when $X = S^4$ and $G = \mathrm{SU}(2)$. We have a principal $\mathrm{SU}(2)$-bundle $P \to S^4$ and associated complex vector bundle $E \to S^4$ of rank 2. Consider an instanton given by an ASD solution A of the duality equations and denote by $\nabla = \nabla_A$ the covariant derivative associated with A.

Consider the twistor bundle $\pi : \mathbb{CP} \quad \to S^4$ and denote by $\tilde{E} := \pi^* E$ the pull-back of the bundle E to $\mathbb{CP}$ via the map π. The anti-selfduality of A implies that its pullback $\tilde{A}$ to the bundle $\tilde{E}$ defines a holomorphic structure on $\tilde{E}$. The obtained holomorphic bundle $\tilde{E} \to \mathbb{CP}$ is by construction holomorphically trivial on j-invariant projective lines in $\mathbb{CP}$ being the fibers of the map π.

Atiyah–Ward theorem. There exists a bijective correspondence between

$$\left\{\begin{matrix}\text{moduli space of} \\ \text{instantons on } S^4\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{holomorphic vector bundles over } \mathbb{CP} \\ \text{which are holomorphically trivial on } \pi\text{-} \\ \text{fibers}\end{matrix}\right\}.$$

There is also a purely complex version of this theorem. Consider it first for the future tube $\mathbb{C}M_+$. Let E be a holomorphic vector bundle over $\mathbb{C}M_+$ and $\nabla = \nabla_A$ is the holomorphic covariant derivative acting on sections of E generated by a holomorphic connection A. This connection is called anti-selfdual (ASD) if its curvature vanishes on all α-planes. The complex variant of Atiyah–Ward theorem asserts that there exists a bijective correspondence between

$$\left\{\begin{matrix}\text{moduli space of holomorphic} \\ \text{ASD-connections on } \mathbb{C}M_+\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{holomorphic vector bundles on} \\ \mathbb{PT}_+ \text{ holomorphically trivial on} \\ \text{projective lines lying in } \mathbb{PT}_+\end{matrix}\right\}.$$

This theorem is based on the following Ward construction. Let $\tilde{E}$ be a holomorphic vector bundle over $\mathbb{PT}_+$ which is holomorphically trivial on projective lines in $\mathbb{PT}_+$. The fiber E_z of the corresponding holomorphic vector bundle $E \to \mathbb{C}M_+$ at $z \quad \mathbb{C}M_+$ consists by definition of holomorphic sections of $\tilde{E}$ over the projective line $\mathbb{CP}^1_z$ corresponding to the point z. If two projective lines $\mathbb{CP}^1_z$ and $\mathbb{CP}^1_{z'}$ intersect, i.e. the points z and z' lie on the same complex light line, we can identify the fibers E_z and $E_{z'}$. In this way we define a parallel transport on E along complex light lines in $\mathbb{C}M_+$ generating a holomorphic connection in E. By construction this connection is anti-selfdual.

For the inverse construction (from E to $\tilde{E}$) it is convenient to use the double diagram

$$\begin{tikzcd} & \mathbb{F}_+ \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ \mathbb{PT}_+ & & \mathbb{C}M_+ \end{tikzcd}$$

where $\mathbb{F}_+$ is the space of $(0,1)$-flags in $\mathbb{PT}_+$, i.e. pairs (point of $\mathbb{PT}_+$, projective line in $\mathbb{PT}_+$ containing this point). The space $\mathbb{C}M_+$ is identified with the Grassmann manifold $G_1(\mathbb{PT}_+)$ of projective lines lying in $\mathbb{PT}_+$, and μ, ν are natural projections. Denote by E' the pull-back of E to a bundle over $\mathbb{F}_+$ via the map ν and by ∇' the pull-back of the connection ∇ to the bundle E'. Define the fibre of the bundle $\tilde{E} \to \mathbb{PT}_+$ at ζ $\mathbb{PT}_+$ as the space of holomorphic sections s' $\Gamma(\mu^{-1}(\zeta), E')$ satisfying the equation $\nabla'_\mu s' = 0$ (∇'_μ is the component of ∇' acting along the fibers of μ). In other words, the fibre $\tilde{E}_\zeta$ consists of horizontal holomorphic sections of E' over $\mu^{-1}(\zeta)$. This definition is correct due to the anti-selfduality of ∇.

The given complex version of Atiyah–Ward theorem remains true if we replace $\mathbb{PT}_+$ by a domain $\tilde{D}$ in $\mathbb{CP}$ such that projective lines in $\tilde{D}$ correspond to the points of some domain D in $\mathbb{C}M$. This domain should have an additional property that the intersection of any complex light line with D is connected and simply connected.

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Quantum Dirichlet forms and their recent applications

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We discuss the notion of classical Dirichlet forms, quadratic forms giving rise to Markov semigroups on the spaces of the form $L^2(X, \mu)$, and its quantum generalisations, defined in terms of von Neumann algebras. Recent applications of quantum Dirichlet forms in the framework of locally compact quantum groups are also outlined.

The originating idea of the classical theory of operator semigroups comes from the desire to describe physical evolutions which are in some sense 'time-invariant', in the sense that what happens to the system between time t and $t + s$ depends only on the time distance s (and the state of the system at time t). In probability such behaviour is usually called the Markov property.

Definition 1. Let X be a Banach space. A C_0-semigroup of operators is a family $(P_t)_{t\geq 0}$ of bounded linear operators on X such that

(i) $P_0 = \mathrm{id}_X$;
(ii) $P_{t+s} = P_t \circ P_s, \quad s, t \geq 0$;
(iii) $\lim_{t\to 0^+} P_t x = x, \quad x \quad X$.

The last property is usually called the *strong* continuity or *point-norm* continuity. Sometimes we need to talk about C_0^**-semigroups*: if Y is a Banach space then $(P_t)_{t\geq 0}$ is called a C_0^*-semigroup on $X = Y^*$ if it is a family of linear weak*-continuous (so also bounded) operators on Y^* such that

$$\lim_{t\to 0^+} (P_t x)(y) = x(y), \qquad x \quad X, y \quad Y.$$

Definition 2. Given a C_0-semigroup of operators $(P_t)_{t\geq 0}$ on X define

$$\mathrm{Dom}(L) := \left\{ x \quad X : \lim_{t\to 0^+} \frac{P_t x - x}{t} \text{ exists} \right\}$$

and further $L : \mathrm{Dom}(L) \to X$ by the obvious formula

$$Lx = \lim_{t\to 0^+} \frac{P_t x - x}{t}, \quad x \quad \mathrm{Dom}(L).$$

We have the following fundamental result (see, for example, [6]).

Theorem 3. *Let $(P_t)_{t\geq 0}$ be a C_0-semigroup of operators on a Banach space X. The map $L : \mathrm{Dom}(L) \to X$ defined above, called the generator of the semigroup $(P_t)_{t\geq 0}$, is a densely defined, closed, linear operator, determining the semigroup uniquely. Further the following conditions are equivalent:*

(i) $\mathrm{Dom}(L) = X$;
(ii) *L is bounded;*
(iii) *$(P_t)_{t\geq 0}$ is* norm continuous *(or* uniformly continuous*), i.e.* $\lim_{t\to 0^+} \|P_t - P_0\| = 0$.

In the latter case,

$$P_t x = \exp(tL)(x) = \sum_{n=0}^{\infty} \frac{(tL)^n x}{n!}, \quad x \quad X.$$

In general the following question is difficult: when is a closed densely defined operator L the generator of a C_0-semigroup?

Theorem 4 (Hille–Yosida). *Let $L : \mathrm{Dom}(L) \to X$ be a linear operator (*$\mathrm{Dom}(L) \subset X$*). The following are equivalent:*

(i) *L is the generator of a C_0-semigroup of contractions (i.e. $\|P_t\| \leq 1, t \geq 0$);*
(ii) *L is closed, densely defined, and for all $\lambda > 0$ we have that the operator $\lambda \mathrm{id}_X - L$ is invertible and*

$$\|\lambda(\lambda \mathrm{id}_X - L)^{-1}\| \leq 1.$$

Replace now X by a Hilbert space H and for each $\xi, \eta \quad \mathsf{H}$ ask about the limits of the form

$$\lim_{t\to 0_+} \left\langle \xi, \frac{\eta - P_t \eta}{t} \right\rangle$$

(which obviously exist for $\eta \quad \mathrm{Dom}(L)$). If further all the operators P_t are self-adjoint, then the usual polarisation identity implies that it suffices to study the densely defined *quadratic* form

$$Q(\xi) := \lim_{t\to 0^+} \left\langle \xi, \frac{\xi - P_t \xi}{t} \right\rangle.$$

Note that then $Q : \mathrm{Dom}(Q) \to \mathbb{R}$. For the results below we refer to [8] and [13].

Theorem 5. *Let H be a Hilbert space. There is a 1–1 correspondence between the following three classes of objects:*

(i) *C_0-semigroups $(P_t)_{t\geq 0}$ of self-adjoint contractions on H;*
(ii) *(unbounded) positive self-adjoint operators A on H;*
(iii) *closed, densely defined quadratic forms Q on H.*

Very roughly speaking, the correspondences are as follows: $-A$ *is the generator of* $(P_t)_{t\geq 0}$*; we have* $P_t = \exp(-tA)$ *(in the sense of the functional calculus for self-adjoint operators), and* $Q(\cdot) = \|A^{\frac{1}{2}} \cdot \|^2$.

Definition 6. Let (Ω, μ) be a space with a (non-negative) measure. A Markov semigroup on $\mathsf{L}^\infty(\Omega, \mu)$ is a C_0^*-semigroup $(P_t)_{t\geq 0}$ on $\mathsf{L}^\infty(\Omega, \mu) = \mathsf{L}^1(\Omega, \mu)^*$ such that

(i) $P_t 1 \leq 1$, $P_t f \geq 0$, $f \quad \mathsf{L}^\infty(\Omega, \mu)_+$, $t \geq 0$;
(ii) $\int_\Omega f d\mu = \int_\Omega P_t f d\mu$, $f \quad \mathsf{L}^\infty(\Omega, \mu)_+$, $t \geq 0$.

Such a semigroup is called symmetric if for all bounded $f, g \quad \mathsf{L}^2(\Omega, \mu)$

$$\int_\Omega \bar{f} P_t g d\mu = \int_\Omega \overline{P_t f} g d\mu.$$

It is called conservative if $P_t 1 = 1$, $t \geq 0$.

All such semigroups restrict/extend to C_0-semigroups of (positivity preserving) contractions on each of the $\mathsf{L}^p(\Omega, \mu)$-spaces for $p \quad [1, \infty)$.

Example 7. Consider the Euclidean space with the Lebesgue measure: $(\mathbb{R}^n, \lambda)$ and define for each $t \geq 0, f \quad \mathsf{L}^\infty(\mathbb{R}^n, \lambda)$

$$(P_t f)(s) = (\ \pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \exp(-\frac{\|s-r\|^2}{t}) f(r) dr, \quad s \quad \mathbb{R}^n.$$

This defines a Markov semigroup – the so-called *heat semigroup* on $\mathbb{R}^n$. In fact, it is a *translation invariant* conservative Markov semigroup, i.e. one of the form

$$P_t f = \mu_t \star f, \quad t \geq 0, f \quad \mathsf{L}^\infty(\mathbb{R}^n, \lambda),$$

where μ_t is a probability measure on $\mathbb{R}^n$.

The generator of the corresponding L^2-semigroup is the *Laplace operator*: the closure of the map $-\Delta$, where

$$(\Delta f)(s) = \sum_{i=1}^n \frac{\partial^2}{\partial s_i^2} f(s_1, \ldots, s_n)$$

for f in the *Schwarz space* $\mathcal{S}(\mathbb{R}^n) \subset \mathsf{L}^2(\mathbb{R}^n, \lambda)$. The corresponding quadratic form is

$$Qf = \sum_{i=1}^n \int_{\mathbb{R}^n} \left| \frac{\partial f}{\partial s_i} \right|^2 ds$$

for $f \quad H^1(\mathbb{R}^n) = \{f \quad \mathsf{L}^2(\mathbb{R}^n) : \frac{\partial f}{\partial s_i} \quad \mathsf{L}^2(\mathbb{R}^n), i = 1, \ldots, n\}$.

Definition 8. Let (Ω, μ) be a space with a (non-negative) measure. Denote by $P_\wedge$ the orthogonal projection onto the closed convex set $\{f \quad \mathsf{L}^2(\Omega, \mu) : 0 \leq f \leq 1\}$. A densely defined closed quadratic form Q on $\mathsf{L}^2(\Omega, \mu)$ is called Dirichlet if for every $f \quad \mathsf{L}^2(\Omega, \mu)_\mathbb{R}$ we have

$$f \quad \mathrm{Dom}(Q) \Longrightarrow P_\wedge f \quad \mathrm{Dom}(Q) \text{ and } Q(P_\wedge f) \leq Q(f).$$

Theorem 9 (Beurling–Deny). *Let (Ω, μ) be a space with a (non-negative) measure. There is a* 1–1 *correspondence between:*

(i) *symmetric Markov semigroups on* $\mathsf{L}^\infty(\Omega, \mu)$*;*
(ii) *Dirichlet forms on* $\mathsf{L}^2(\Omega, \mu)$*,*

If the measure μ is finite, then the Markov semigroup in question is conservative if and only if $Q(1_\Omega) = 0$.

We can choose whether we prefer to work with real or complex $\mathsf{L}^2(\Omega, \mu)$. The closedness condition can be replaced by lower semicontinuity, and with forms defined everywhere, but sometimes taking value $+\infty$.

Let G be a locally compact group. A family of probability measures $(\mu_t)_{t\geq 0}$ on G is called a *convolution semigroup* if we have $\mu_0 = \delta_e$, $\mu_{t+s} = \mu_t \star \mu_s, s, t \geq 0$ and $\int_G f d\mu_t \overset{t\to 0^+}{\longrightarrow} f(e)$ for all $f \quad C_b(G)$.

Theorem 10. *Let G be a locally compact group (with a fixed left Haar measure denoted dg). Then there is a* 1–1 *correspondence between the following classes of objects:*

(i) *translation invariant symmetric conservative Markov semigroups on the measure space (G, dg);*
(ii) *translation invariant Dirichlet forms on* $\mathsf{L}^2(G, dg)$ *(modulo a scalar perturbation);*
(iii) *convolution semigroups of symmetric probability measures on G;*
(iv) *symmetric Lévy processes on G, that is G-valued stochastic processes indexed by $\mathbb{R}^+$ with independent, symmetrically and identically distributed increments.*

Everywhere above we work with left translations (one can of course consider the right-handed version of the result). Note that the maps P_t as above, given by the prescription

$$(P_t f)(s) = \int_G f(r^{-1}s) d\mu_t(r),$$

map continuous bounded functions into continuous bounded functions: this is usually called the *Feller property* and is of big importance in classical probability.

We will now present some of the earlier ideas in the quantum setting. We will first replace the space (Ω, μ) by the algebra $\mathsf{L}^\infty(\Omega, \mu)$, and then consider general, not necessarily commutative algebras which 'look like' $\mathsf{L}^\infty(\Omega, \mu)$ – namely the von Neumann algebras.

Definition 11. A von Neumann algebra M is a weak*-closed unital *-subalgebra of the algebra $B(\mathsf{H})$ for some Hilbert space H (equivalently: a *-subalgebra $\mathsf{M} \subset B(\mathsf{H})$ such that $\mathsf{M} = \mathsf{M}''$ – the algebra is equal to its bicommutant). We say that $\varphi : \mathsf{M}_+ \to [0, \infty]$ is a normal semifinite faithful weight on M, where M_+ denotes the positive cone of M, if φ is a homogeneous, additive map such that

(i) $\mathfrak{n}_\varphi = \{x \quad \mathsf{M} : \varphi(x^*x) < \infty\}$ is weak*-dense in M (semifiniteness);

(ii) when $x_i \nearrow x$, then $\varphi(x) \leq \limsup_{i\in\mathcal{I}} \varphi(x_i)$ (lower semicontinuity/normality);

(iii) $\varphi(x^*x) = 0$ implies $x = 0$ (faithfulness).

We call such a weight a state if $\varphi(1) = 1$. Weights extend to linear functionals on $\mathfrak{m}_\varphi = \operatorname{span}\{x \quad \mathsf{M}_+ : \varphi(x) < \infty\}$; so normal faithful states can be viewed as a subclass of usual bounded functionals on M. Finally, φ as above is called tracial if for all $x, y \quad \mathfrak{n}_\varphi$ we have $\varphi(xy) = \varphi(yx)$.

Example 12. Consider the following examples:

(i) $\mathsf{M} = \mathsf{L}^\infty(\Omega, \mu) \subset B(\mathsf{L}^2(\Omega, \mu))$, $\varphi(f) = \int f d\mu$;

(ii) $\mathsf{M} = M_n = B(\mathbb{C}^n)$ (the algebra of n by n complex matrices), $\varphi = \frac{1}{n}\mathrm{Tr}$ (tracial state), or $\varphi(\cdot) = \mathrm{Tr}\,(D\cdot)$, where D is a *density matrix*, that is a positive-definite matrix of trace 1;

(iii) $\mathsf{M} = B(\ell^2)$, $\varphi(\cdot) = \mathrm{Tr}\,(D\cdot)$, where D is a density matrix (a positive trace class operator of trace 1), which yields a non-tracial state; or $\varphi = \mathrm{Tr}$ – which yields a tracial weight;

(iv) G-discrete group, $\mathsf{H} = \ell^2(G)$. For $g \quad G$ let $\lambda_g \quad B(\ell^2(G))$ be a (left) shift operator: $\lambda_g(\delta_h) = \delta_{gh}$, $h \quad G$. Then define $\mathsf{M} = \mathrm{VN}(G) = \{\lambda_g : g \quad G\}'' \subset B(\ell^2(G))$. The canonical tracial state on $\mathrm{VN}(G)$ is $\varphi = \omega_{\delta_e}$, i.e. $\varphi(x) = \langle \delta_e, x\delta_e\rangle, x \quad \mathrm{VN}(G)$. The construction of $\mathrm{VN}(G)$ generalises to the situation where G is an arbitrary locally compact group, with φ becoming the so-called *Plancherel weight*. If G is abelian, we have $\mathrm{VN}(G) = \mathsf{L}^\infty(\widehat{G})$ and the Plancherel weight of G is simply a Haar measure of $\widehat{G}$.

Given a map $\Phi : \mathsf{M} \to \mathsf{M}$ and $n \quad \mathbb{N}$ we can always define 'entrywise' a map $\Phi^{(n)} : \mathsf{M} \otimes M_n \to \mathsf{M} \otimes M_n$, where $\mathsf{M} \otimes M_n$ is the von Neumann algebra identified as the algebra of n by n matrices with entries in M. A map Φ as above is called *positive* if $\Phi(\mathsf{M}_+) \subset \mathsf{M}_+$, and *completely positive* if each $\Phi^{(n)}$ is positive.

Definition 13. Let (M, φ) be as above. A quantum Markov semigroup is a C_0^*-semigroup of normal maps $(P_t)_{t\geq 0}$ on $\mathsf{M} = (\mathsf{M}_*)^*$ such that

(i) $P_t 1 \leq 1$, and each P_t is completely positive ($t \geq 0$);

(ii) $\varphi(f) = \varphi(P_t f)$, $f \quad \mathsf{M}_+$, $t \geq 0$.

The symmetry condition becomes in general more complicated! We can associate to a pair (M, φ) non-commutative L^p-spaces, but the way of doing this is non-trivial.

If φ is tracial, the procedure is simpler. We can just consider

$$\mathfrak{m}^{(p)} := \{x \quad \mathsf{M} : \varphi(|x|^p) < \infty\}, \quad p \quad [1, \infty),$$

so that for example $\mathfrak{n}_\varphi = \mathfrak{m}^{(2)}$, and complete it with respect to the norm

$$\|x\|_p = \varphi(|x|^p)^{\frac{1}{p}}.$$

However, when φ is not tracial, this is not a norm!

There are several constructions in the non-tracial case, we will use the one due to Haagerup (see [11] and [15]), based on the Tomita–Takesaki theory, concerning the behaviour of the non-tracial states or weights. We will just list some properties of the resulting Banach spaces:

- $\mathsf{L}^p(\mathsf{M},\varphi)$ are certain spaces of (unbounded) operators on a larger Hilbert space $\mathcal{H}$, closed under taking adjoints and positive parts;
- we have natural isomorphisms $\mathsf{L}^\infty(\mathsf{M},\varphi) \approx \mathsf{M}$, $\mathsf{L}^1(\mathsf{M},\varphi) \approx \mathsf{M}_*$;
- different spaces have trivial intersections, for example $\mathsf{L}^\infty(\mathsf{M},\varphi) \cap \mathsf{L}^2(\mathsf{M},\varphi) = \{0\}$;
- there are different ways of getting from M into $\mathsf{L}^2(\mathsf{M},\varphi)$. Tomita–Takesaki theory allows us in a sense to write always $\varphi(\cdot) = \mathrm{Tr}(D\cdot)$, where D is a certain 'density-like' operator. Symbolically we may describe the *GNS-embedding* $x \mapsto xD^{\frac{1}{2}}$ and the *KMS-embedding* as $x \mapsto D^{\frac{1}{4}}xD^{\frac{1}{4}}$. We will denote the latter by $\iota^{(2)} : \mathfrak{n}_\varphi \to \mathsf{L}^2(\mathsf{M},\varphi)$.

All that originates from the automorphism group $(\sigma_t)_{t\in\mathbb{R}}$ acting on M, the so-called *modular automorphism group*, ruling the non-traciality of φ:

$$\varphi(xy) = \varphi(y\sigma_i(x)),$$

for 'good' $x, y \quad \mathsf{M}$; $\sigma_i(x)$ is defined via a suitable holomorphic extension of the function $t \mapsto \sigma_t(x)$. We have in fact (viewing all the operators as acting on the large Hilbert space $\mathcal{H}$)

$$\sigma_t(x) = D^{it}xD^{-it}, \quad x \quad \mathsf{M}, t \quad \mathbb{R}.$$

Indeed, consider the following informal computation:

$$\begin{aligned}\varphi(xy) &= \mathrm{Tr}\,(Dxy) = \mathrm{Tr}\,(yDx) = \mathrm{Tr}\,(y(DxD^{-1})D) = \mathrm{Tr}\,(Dy(D^{-1}xD)) \\ &= \varphi(y(D^{-1}xD)) = \varphi(y\sigma_i(x))\end{aligned}$$

Definition 14. A quantum Markov semigroup $(P_t)_{t\geq 0}$ on (M,φ) is said to be KMS-symmetric if for each $t \geq 0$ the prescription

$$P_t^{(2)}(\iota^{(2)}(x)) = \iota^{(2)}(P_t x), \quad x \quad \mathfrak{n}_\varphi,$$

is well-defined and yields a bounded self-adjoint operator on $\mathsf{L}^2(\mathsf{M},\varphi)$.

Example 15. If $(\mathsf{M},\varphi) = (\mathsf{L}^\infty(\Omega,\mu), \int \cdot d\mu)$, then quantum Markov semigroups on (M,φ) are precisely the Markov semigroups on (Ω,μ) discussed above.

Example 16. Let G be again a discrete group, $\mathsf{M} = \mathrm{VN}(G)$, φ–canonical trace. Suppose that $\psi : G \to \mathbb{R}$ is a conditionally negative definite symmetric function, i.e. a function such that

(i) $\forall_{g\in G} \quad \psi(g) = \psi(g^{-1})$;
(ii) $\forall_{n\in\mathbb{N}}\forall_{\lambda_1,\ldots,\lambda_n\in\mathbb{C}}\forall_{g_1,\ldots,g_n\in G} \quad \sum_{i=1}^n \lambda_i = 0 \Longrightarrow \sum_{i,j=1}^n \overline{\lambda_i}\lambda_j\psi(g_i^{-1}g_j) \leq 0.$

Then the family of maps $(P_t)_{t\geq 0}$ on $\mathrm{VN}(G)$ given by the formulas

$$P_t(\lambda_g) = \exp(-t\psi)\lambda_g, \quad g \quad G, t \geq 0,$$

forms a quantum Markov semigroup of *Herz–Schur multipliers.*

Example 17. If $(\mathsf{M}, \varphi) = (M_n, \mathrm{tr})$, then every quantum Markov semigroup on (M, φ) is norm continuous and we can in fact characterise the generators (see [5]):

$$P_t x = \exp(tL)x, \quad x \quad M_n, t \geq 0,$$

with L of the *Lindblad* or *Gorini–Kossakowski–Sudarshan form*:

$$Lx = -i[H, x] + \frac{1}{2} \sum_{\alpha} \left([V_\alpha x, V_\alpha^*] + [V_\alpha, x V_\alpha^*] \right), \quad x \quad M_n.$$

Here $H = H^* \quad M_n$, $V_\alpha \quad M_n$, $\sum_\alpha [V_\alpha, V_\alpha^*] = 0$, and $[A, B] := AB - BA$ are the *commutators.*

We are ready to discuss the Dirichlet forms in the quantum context.

Definition 18. Let (M, φ) be as above. Denote by $P_\wedge$ the orthogonal projection onto the closed convex set $\left\{ f \quad \mathsf{L}^2(\mathsf{M}, \varphi) : 0 \leq f \leq D^{\frac{1}{2}} \right\}$. A densely defined closed quadratic form Q on $\mathsf{L}^2(\mathsf{M}, \varphi)$ is called Dirichlet if for every $f \quad \mathsf{L}^2(\mathsf{M}, \varphi)_{\mathbb{R}}$ we have

$$f \quad \mathrm{Dom}(Q) \Longrightarrow P_\wedge f \quad \mathrm{Dom}(Q) \text{ and } Q(P_\wedge f) \leq Q(f).$$

The form Q as above is called completely Dirichlet if for every $n \quad \mathbb{N}$ the natural associated quadratic form on $\mathsf{L}^2(\mathsf{M} \otimes M_n, \varphi \otimes \mathrm{tr}_n)$ is Dirichlet.

Theorem 19 ([3, 9, 10, 14]). *Let* (M, φ) *be as above. There is a* 1–1 *correspondence between:*

(i) *quantum KMS-symmetric Markov semigroups on* (M, φ)*;*
(ii) *completely Dirichlet forms on* $\mathsf{L}^2(\mathsf{M}, \varphi)$*.*

If φ *is a state, then the quantum Markov semigroup in question is conservative if and only if its corresponding completely Dirichlet form* Q *satisfies* $Q(D^{\frac{1}{2}}) = 0$*.*

We finally connect the latter results to the world of locally compact quantum groups. A locally compact quantum group $\mathbb{G}$ (à la Kustermans–Vaes, see [12]) is a virtual object, studied via a von Neumann algebra $\mathsf{L}^\infty(\mathbb{G})$, equipped with the *coproduct* (carrying all the information about $\mathbb{G}$)

$$\Delta : \mathsf{L}^\infty(\mathbb{G}) \to \mathsf{L}^\infty(\mathbb{G}) \overline{\otimes} \mathsf{L}^\infty(\mathbb{G})$$

and a canonical *right Haar weight* ϕ. We can also study the associated (reduced) C^*-object $\mathrm{C}_0(\mathbb{G})$ and its universal version $\mathrm{C}_0^u(\mathbb{G})$, with the counit: a character $\epsilon : \mathrm{C}_0^u(\mathbb{G}) \to \mathbb{C}$. By the analogy with the classical situation we write $\mathsf{L}^2(\mathbb{G})$ for the GNS Hilbert space of the right invariant Haar weight ϕ on $\mathsf{L}^\infty(\mathbb{G})$.

Each LCQG $\mathbb{G}$ admits the dual LCQG $\widehat{\mathbb{G}}$, and we have a canonical isomorphism $\mathsf{L}^2(\mathbb{G}) \approx \mathsf{L}^2(\widehat{\mathbb{G}})$. In particular for a standard locally compact group G we have

$$\mathsf{L}^\infty(\widehat{G}) = \mathrm{VN}(G), \quad \mathrm{C}_0(\widehat{G}) = \mathrm{C}^*(G), \quad \mathrm{C}_0^u(\widehat{G}) = \mathrm{C}^*(G).$$

Definition 20. A family $(\mu_t)_{t \geq 0_+}$ of states on $\mathrm{C}_0^u(\mathbb{G})$ is called a convolution semigroup of states if

(i) $\mu_{t+s} = \mu_t \star \mu_s := (\mu_t \otimes \mu_s) \circ \Delta, \quad t, s \geq 0;$

(ii) $\mu_t(a) \overset{t\to 0^+}{\longrightarrow} \mu_0(a) := \epsilon(a), \quad a \quad \mathrm{C}_0^u(\mathbb{G})$.

The algebra $\mathrm{C}_0^u(\mathbb{G})$ admits a canonical involutive operator R^u, so called *universal unitary antipode* (playing the role of the inverse operation). The two next theorems come from [14]; for compact quantum groups they were proved in [4].

Theorem 21. *Let* $\mu \quad S(\mathrm{C}_0^u(\mathbb{G}))$. *The associated operator* $R_\mu : \mathsf{L}^\infty(\mathbb{G}) \to \mathsf{L}^\infty(\mathbb{G})$ *(which can be informally thought of as the map* $(\mu \otimes \mathrm{id}) \circ \Delta$*) is unital, completely positive,* ϕ*-preserving. The map* R_μ *is KMS-symmetric iff* $\mu = \mu \circ \mathsf{R}^u$. *Its KMS implementation (acting on* $\mathsf{L}^2(\mathbb{G})$*) is always bounded and belongs to* $\mathsf{L}^\infty(\widehat{\mathbb{G}})$.

We are ready to present the main result connecting the notions presented earlier.

Theorem 22. *Let* $\mathbb{G}$ *be a locally compact quantum group. There exist* 1–1 *correspondences between:*

(i) *convolution semigroups* $(\mu_t)_{t\geq 0}$ *of* R^{u}*-invariant states of* $\mathrm{C}_0^u(\mathbb{G})$*;*

(ii) C_0^**-semigroups* $(T_t)_{t\geq 0}$ *of normal, unital, completely positive maps on* $\mathsf{L}^\infty(\mathbb{G})$ *that are KMS-symmetric with respect to* ϕ *and satisfy* $\Delta \circ T_t = (T_t \otimes \mathrm{id}) \circ \Delta$ *for every* $t \geq 0$*;*

(iii) *completely Dirichlet forms* Q *on* $\mathsf{L}^2(\mathbb{G})$ *with respect to* ϕ *that are invariant under* $\mathcal{U}(\mathsf{L}^\infty(\hat{\mathbb{G}})')$ *(modulo a scalar perturbation).*

We finish by stating two applications of the above to geometric properties of quantum groups, also proved in [14]. Relevant definitions can be found in [1,2] and [7].

Theorem 23. *Let* $\mathbb{G}$ *be a second countable locally compact quantum group. Then* $\widehat{\mathbb{G}}$ *has Property (T) if and only if every convolution semigroup of* R^u*-invariant states on* $\mathrm{C}_0^u(\mathbb{G})$ *has a bounded generator.*

Theorem 24. *Let* $\mathbb{G}$ *be a second countable locally compact quantum group. Then* $\widehat{\mathbb{G}}$ *has the Haagerup property if and only if there exists a convolution semigroup of* R^u*-invariant states on* $\mathrm{C}_0^u(\mathbb{G})$ *such that the* L^2*-implementations of the associated convolution operators, acting on* $\mathsf{L}^2(\mathbb{G})$*, in fact belong to* $\mathrm{C}_0(\widehat{\mathbb{G}})$.

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Lagrangian approach to Geometric Quantization

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The Lagrangian approach to Geometric Quantization is based on realizing Lagrangian submanifolds of a given classical phase space as quantum states. Geometrically it is clear how the corresponding quantum dynamics can be derived, but the necessary measurement process cannot be defined unless one introduces some additional data (e.g. half-weights on Lagrangian submanifolds). Here we sketch a possible way to do so, when special Bohr–Sommerfeld geometry can be exploited.

Geometric Quantization. Every symplectic manifold (M, ω) can be regarded as the phase space of some classical mechanical system, see [1]. Namely the points of M are understood as the classical states, and any real smooth function f on M is an observable of the system. The Poisson algebra $\mathcal{P} = (C^\infty(M, \mathbb{R}), \{\cdot, \cdot\}_\omega)$ determines our symplectic manifold (M, ω) uniquely.

Quantization is a procedure which presents a representation of the Poisson algebra $\mathcal{P}$ in the Lie algebra of self adjoint operators on a Hilbert space $\mathcal{H}$; from the geometric viewpoint it is given by a functor $q : (M, \omega) \to \mathcal{H}$ such that a classical observable $f \quad C^\infty(M, \mathbb{R})$ goes to a self-adjoint operator $q(f) \quad Op(\mathcal{H})$ in such a way that (1) the correspondence is linear, (2) the Poisson brackets go to the commutator, and (3) the representation is irreducible.

There are many approaches to the Problem of Quantization, but we are mostly interested in the approaches where the geometry of the phase space gives the ingredients of the quantization. This is called *Geometric Quantization*, see [2].

Geometric formulation of Quantum Mechanics. Every Hilbert space $\mathcal{H}$ can be projectivized $\mathcal{H} \mapsto \mathbb{P}\mathcal{H}$; the hermitian scalar product on $\mathcal{H}$ generates the standard Kähler structure of Hodge type on the projective space $\mathbb{P}\mathcal{H}$, and all axioms of

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QM can be reformulated in the language of Kähler geometry, see [3, 4]. Every self-adjoint operator $\hat{F}$ is transformed into a smooth function $f(\hat{F}) \quad C^{\infty}(\mathbb{P}\mathcal{H}, \mathbb{R})$ which satisfies the following condition: its Hamiltonian vector field preserves the Kähler structure. Such functions are called *Berezin symbols* after F. Berezin.

Algebraic–geometric quantization. Combining a Geometric formulation of Quantum Mechanics and the Quantization problem we can propose the following generalization of the latter, see [4]. Let (M, ω) be a symplectic manifold understood as the phase space of certain classical mechanical system. We would like to find a Kähler manifold $\mathcal{K}$ endowed with Kähler structure (Ω, I, G) such that Ω is a symplectic form, I is a compatible integrable complex structure and G is the corresponding Riemann metric, together with a linear map $q : C^{\infty}(M, \mathbb{R}) \to C^{\infty}_q(\mathcal{K}, \mathbb{R})$ such that $q(1) = 1$ and $q(\{f_1, f_2\})_{\omega} = \{q(f_1), q(f_2)\}_{\Omega}$.

As an example we can present the Berezin quantization method, when (M, ω) itself is a Kähler manifold, and one quantizes only the functions which are Berezin symbols.

Lagrangian approach to Geometric Quantization. In the Lagrangian approach one studies Lagrangian submanifolds of the phase space (M, ω) understood as quantum states. The starting point is given by the WKB method to get solutions of the Schrödinger equation for a given density. In the generic situation when (M, ω) is topologically non trivial, one has the following interesting parallel to the standard QM. Namely each smooth function f on (M, ω) induces the corresponding Hamiltonian flow $\phi^t_{X_f}$ which preserves the Lagrangian condition, so if $S \subset M$ is a Lagrangian submanifold then $\phi^t_{X_f}(S)$ is Lagrangian as well. Therefore for the space of all Lagrangian submanifolds $\mathcal{L}$ every smooth function can be regarded as an operator which generates certain dynamics. Moreover, we have the following alternative: this operator can be measured on a state S if and only if S is a stationary point of this action. Indeed, it is a simple fact: the Hamiltonian vector field X_f is tangent to the Lagrangian submanifold S if and only if f is constant when restricted to S (thus this constant can be taken as the result of the measurement process). It is similar to the basic fact from the standard QM: a quantum state is an eigenstate of a self-adjoint operator if and only if it is a stationary point of the corresponding evolution generated by this operator.

Bohr–Sommerfeld geometry. A Hamiltonian flow preserves the *periods* of any Lagrangian submanifold $S \subset M$ therefore, instead of the space $\mathcal{L}$ of all Lagrangian submanifolds, it is much more convenient to consider the subspace of Bohr–Sommerfeld Lagrangian submanifolds. Suppose first that our given symplectic manifold (M, ω) is simply connected and that the symplectic form ω is integer, so the cohomology class $[\omega]$ belongs to the lattice $H^2(M, \mathbb{Z})$. In the literature one often says that then (M, ω) satisfies the Bohr–Sommerfeld condition for manifolds. In this case one says that a Lagrangian submanifold $S \subset M$ is Bohr–Sommerfeld if for any loop $\gamma \subset S$ and any disc$B_2 \subset M$ such that $\partial B_2 = \gamma$ one has $\int_{B_2} \omega \quad \mathbb{Z}$. In [5] we constructed $\mathcal{B}_S$ – *moduli space of Bohr–Sommerfeld Lagrangian submanifolds of fixed topological type*: it is an infinite-dimensional real smooth Fréchet manifold. For any function $f \quad C^\infty(M,\mathbb{R})$ the induced Hamiltonian flow $\phi^t_{X_f}$ gives a one-parameter family of automorphisms of $\mathcal{B}_S$, and we have a description of the corresponding tangent vector $\Theta(f) \quad T_S\mathcal{B}_S$ for each point $S \quad \mathcal{B}_S$.

ALAG – program. Problems. Thus as we have seen the moduli space $\mathcal{B}_S$ is a good and natural candidate to be a space of "quantized states" but dynamically only. In QM we must have an appropriate measurement process, see [6], and to restore this part of the QM setting one has to "complexify" the moduli space $\mathcal{B}_S$ since in the Algebraic–Geometric Quantization problem we need a Kähler manifold. In [5] exactly this problem was partially solved: we constructed a "half-weighted" version of the moduli space $\mathcal{B}_S$. Namely as the result of the ALAG procedure applied to a simply connected compact symplectic manifold (M,ω) with integer symplectic form one gets the moduli space of half-weighted Bohr–Sommerfeld Lagrangian submanifolds of fixed topological type and volume $\mathcal{B}_S^{hw,}$ – an infinite-dimensional almost-Kähler smooth manifold fibered over $\mathcal{B}_S$. The main problem of the construction is that the complex structure constructed on $\mathcal{B}_S^{hw,}$ is not integrable, and the so called BPU map is not holomorphic. However a realization of Algebraic–Geometric Quantization was given in this set up: for any function $f \quad C^\infty(M,\mathbb{R})$ the corresponding smooth Berezin symbol $q(f) \quad C^\infty(\mathcal{B}_S^{hw,} \ ,\mathbb{R})$ is presented by the very simple formula $q(f)(S,\theta) = \tau \int_S f|_S \theta^2$, see [4].

Complexification. Now we can pose the "problem of complexification" for the "pure" moduli space $\mathcal{B}_S$: we would like to find a Kähler manifold $\mathcal{K}$ such that (1) $\mathcal{B}_S$ is a "real part" of $\mathcal{K}$, (2) the natural action of a real smooth $f \quad C^\infty(M,\mathbb{R})$ on $\mathcal{B}_S$ described above can be "lifted" to a Hamiltonian action on this $\mathcal{K}$ by Kähler isometries, and it should imply the existence of correspondence $q : C^\infty(M,\mathbb{R}) \to C^\infty_q(\mathcal{K},\mathbb{R})$ thus this "complexification" should lead to a solution to the Algebraic–Geometric Quantization problem. "Real part" means that either $\mathcal{B}_S$ is embedded into $\mathcal{K}$ as a totally real submanifold or it is fibered over $\mathcal{B}_S$ with Lagrangian fibers.

Special Bohr–Sommerfeld geometry. The new construction presented in [7] can be exploited in the "complexification problem": for a given simply connected compact symplectic manifold (M,ω) with integer symplectic form we construct a certain cycle $\mathcal{U}_{SBS}$ in the direct product $\mathcal{B}_S \times \mathbb{P}\Gamma(M,L)$ where $\Gamma(M,L)$ is the space of smooth sections of the prequantization line bundle $L \to M$ uniquely determined by the condition $c_1(L) = [\omega]$. A pair (S,p) belongs to $\mathcal{U}_{SBS}$ if the Bohr–Sommerfeld Lagrangian submanifold $S \subset M$ is special with respect to the smooth section α, at the point $p \quad \mathbb{P}\Gamma(M,L)$ (the definitions can be found in [7]). On the other hand, we proved that $\mathcal{U}_{SBS}$ is *weakly Kähler* in [7]. But the whole space $\mathcal{U}_{SBS}$ is too big to be a "complexification" of $\mathcal{B}_S$; however it is fibered over $\mathcal{B}_S$ by its very definition. A natural idea arises: to find an appropriate subspace $\mathcal{K} \subset \mathcal{U}_{SBS}$ such that (1) $\mathcal{K}$ is a complex subspace, (2) the natural projection $q : \mathcal{U}_{SBS} \to \mathcal{B}_S$ is reduced to a nice projection $\tilde{q} : \mathcal{K} \to \mathcal{B}_S$ with Lagrangian fibers.

Possible solution: first step. In the recent preprint [8] we present several remarks that could lead to the construction of an appropriate $\mathcal{K} \subset \mathcal{U}_{SBS}$. Namely, the canonical projection $q : \mathcal{U}_{SBS} \to \mathcal{B}_S$ splits as a combination of the following two maps. The first map is given by the representation of smooth sections from $\Gamma(M, L)$ by complex 1-forms on the complement $M \backslash D$ where D is the zero set of the sections, see [7, 8]. If α $\Gamma(M, L)$ then the 1-form is given by the formula $\rho(\alpha) = \frac{1}{2\pi} \frac{\nabla_a \alpha}{\alpha}$, and the main properties are the following: 1) the real part $\mathrm{Re}\rho(\alpha)$ is an exact form, 2) if S is α-special Bohr–Sommerfeld then $\mathrm{Im}\, \rho(\alpha)|_S \equiv 0$. This implies that any pair (S, p) $\mathcal{U}_{SBS}$ can be naturally sent to a point in the tangent bundle $T\mathcal{B}_S$, namely one takes the restriction $\rho(\alpha)|_S$ and since it is a real exact 1-form then it represents a tangent vector from $T_S\mathcal{B}_S$. Denoting this map as $\tau : \mathcal{U}_{SBS} \to T\mathcal{B}_S$ one easily deduces that the canonical projection q splits as $\pi \circ \tau$ where $\pi : T\mathcal{B}_S \to \mathcal{B}_S$ is the canonical projection to the base. Thus if we find an appropriate complex section $\sigma : T\mathcal{B}_S \hookrightarrow \mathcal{U}_{SBS}$ of the fibration $\tau : \mathcal{U}_{SBS} \to T\mathcal{B}_S$ then the image $\mathcal{K} = \sigma(T\mathcal{B}_S)$ would be the desired complex Kähler manifold fibered over $\mathcal{B}_S$.

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